流体数学セミナー
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二流体問題の有限要素近似とその数値シミュレーション

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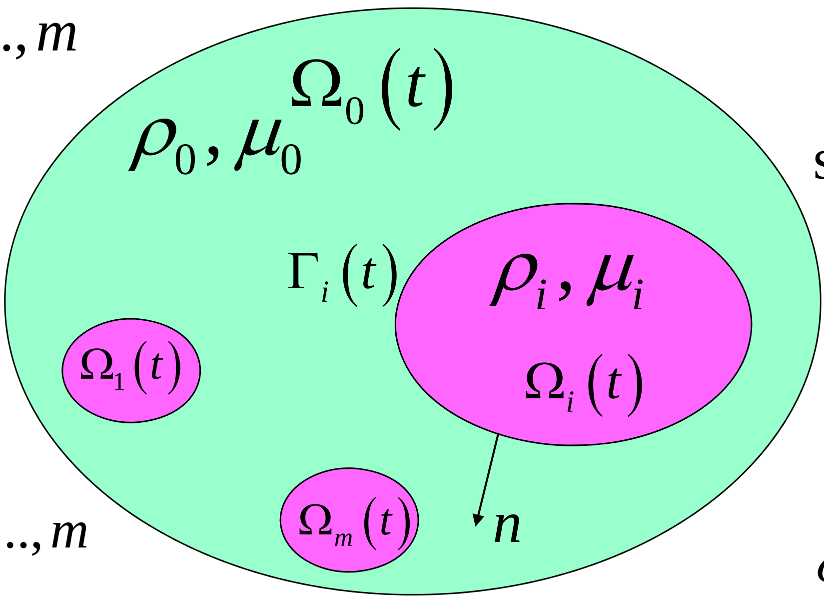
Mathematical Formulation of Multi-Phase Flow Problems

Multiphase flow problems with interface tension(1)

Fluids $0, 1, \ldots, m$

NS eqns



slip BC or $u = 0$

$i = 1, \ldots, m$

κ : curvature
σ_i : coefficient

Interface condition: $[u]_{\Gamma_i} = 0, \; \left[-pn + 2\mu D(u) n\right]_{\Gamma_i} = \sigma_i \kappa n$

Note. $\bar{\Omega}_i(t) \subset \Omega \quad (i = 1, \ldots, m)$ $\qquad D(u) \equiv \frac{1}{2}\left(\nabla u^T + \left(\nabla u^T\right)^T\right)$

Multiphase flow problems with interface tension(2)

$\left(u, p\right)$ satisfies NS eqns. in each domain $\Omega_k\left(t\right)$, $\forall k = 0,\ldots,m$

$$\rho_k\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) - \nabla\left(2\mu_k D(u) - pI\right) = \rho_k f$$
$$\nabla\cdot u = 0$$

Interface conditions on $\Gamma_i\left(t\right) \equiv \bar{\Omega}_0\left(t\right)\cap\bar{\Omega}_i\left(t\right)$

$$\left[u\right] = 0,\ \left[-pn + 2\mu D\left(u\right)n\right]_{\Gamma_i} = \sigma_i \kappa n$$

Boundary conditions on $\Gamma\times\left(0,T\right)$, $\Gamma\equiv\partial\Omega$

$$u\cdot n = 0,\ D\left(u\right)n\times n = 0\ \left(\text{or}\quad u = 0\right)$$

Evolution of $\Gamma_i\left(t\right)\equiv\left\{\chi_i\left(s,t\right); s\in[0,1)\right\}$

$$\frac{\partial\chi_i}{\partial t} = u\left(\chi_i, t\right),\ i = 1,\cdots,m$$

Initial conditions at $t = 0$

$$u = u^0,\quad \Omega_k\left(0\right) = \Omega_k^0,\ \forall k = 0,\ldots,m$$

A weak formulation for MPF problems (1)

$$X \equiv \left\{\chi\in\left(H^1\left(0,1\right)^2\right)^m;\ \chi\left(1\right) = \chi\left(0\right)\right\}$$
$$V \equiv \left\{v\in H^1(\Omega)^2; v\cdot n = 0\left(x\in\Gamma\right)\right\},\quad Q\equiv L_0^2\left(\Omega\right)$$

Find $(\chi, u, p):(0,T)\to X\times V\times Q$ such that

$$\frac{\partial\chi}{\partial t} = u\left(\chi, t\right)$$
$$\left(\rho\frac{\partial u}{\partial t} + \frac{1}{2}u\frac{\partial\rho}{\partial t}, v\right) + a_1(\rho,u,u,v) + a_0(\rho,u,v) + b(v,p) = \left(\rho f, v\right) - d\left(\chi, v\right)$$
$$b(u,q) = 0$$
$$\left(\forall(v,q)\in V\times Q\right)$$

A weak formulation for MPF problems (2)

where

$$a_1(\rho,w,u,v) \equiv \int_\Omega \frac{1}{2}\rho\Big(\big[(w\cdot\nabla)u\big]\cdot v - \big[(w\cdot\nabla)v\big]\cdot u\Big)dx$$

$$a_0(\rho,u,v) \equiv \int_\Omega 2\mu(\rho)D(u):D(v)dx$$

$$b(v,q) \equiv -\int_\Omega (\nabla\cdot v)q\,dx$$

$$d(u,v) \equiv \sum_{i=1}^{m} \int_{\Gamma_i} \sigma_i \frac{du}{d\ell}\cdot\frac{dv}{d\ell}d\ell = \sum_{i=1}^{m} \int_0^1 \sigma_i \frac{du}{ds}\cdot\frac{dv}{ds}\left|\frac{\partial\chi_i}{\partial s}\right|^{-1} ds$$

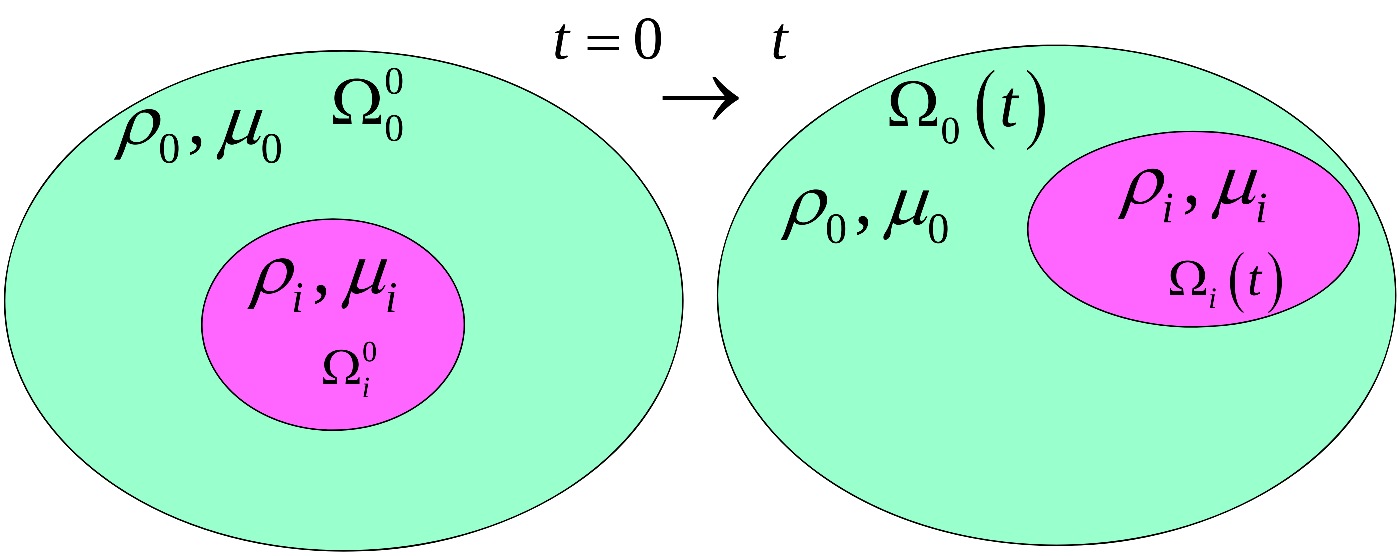
$$\mu = \mu(\rho) \in L^\infty$$

$\rho \in L^\infty(\Omega\times(0,T))$: a function determined from $\{\chi\}$

i.e. $\rho = \rho(\chi)$

Multiphase fluid flow problems

Fluid *i*



$$\frac{\partial\rho}{\partial t} + u\cdot\nabla\rho = 0$$

$$\rho(x,0) = \rho^0(x) \equiv \rho_i \quad (x \in \Omega_i^0)$$

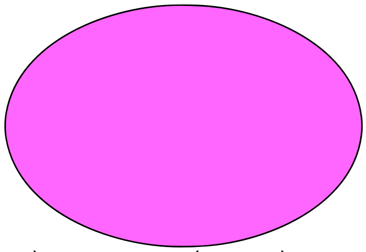
$$\Omega_i(t) \equiv \{\, x\in\Omega; \rho(x,t) = \rho_i \,\},\ \ i = 0,\ldots,m$$

An interface tracking method

$$\chi_i : [0,1) \times (0,T) \to \Re^2$$

$$\Gamma_i(t) \equiv \partial\Omega_i(t) = \left\{ \chi_i(s,t); s \in [0,1) \right\},\ t \in (0,T)$$

$$\frac{\partial \chi}{\partial t} = u(\chi, t)$$



$$\chi(\cdot, t) \equiv \chi_i(\cdot, t)$$

Note. Tryggvason et al. [2001], FDM, no stability arguments

Note. $\leftrightarrow$ Interface capturing method
signed distance function, density function

Interface Tracking Energy-Stable Finite Element Scheme

Finite Element Scheme P0/P2/P1

- Finite element approximation

$$\rho_h^n \in \Psi_h : P_0 - \text{element},$$

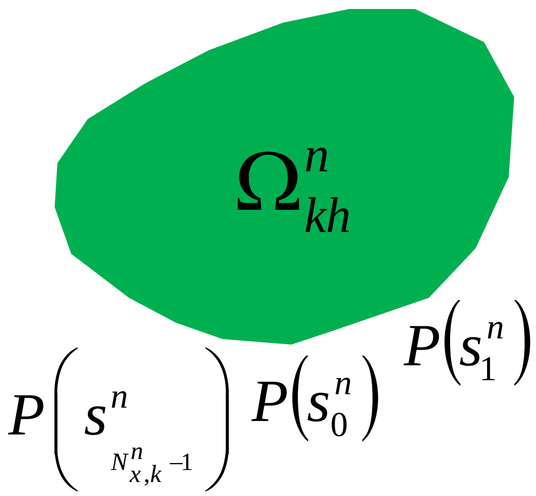
$$u_h^n \in V_h : P_2 - \text{element},$$

$$p_h^n \in Q_h : P_1 - \text{element}$$

Approximate spaces for the interface

$$X_h \subset X \equiv \left\{ \chi \in \left(H^1(0,1)^2 \right)^m ; \chi(1) = \chi(0) \right\}$$

$$\chi_h^n \in X_h$$



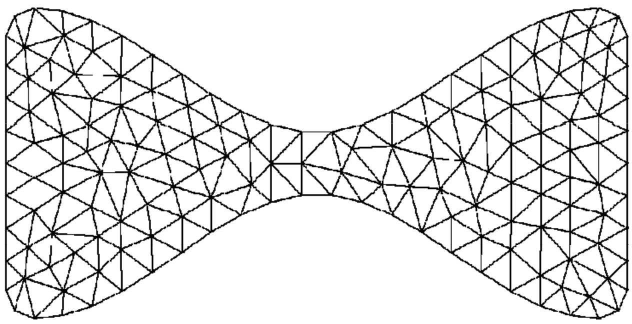
$$0 \equiv s_0^n < s_1^n < \cdots < s_{N_{x,k}^n}^n \equiv 1$$

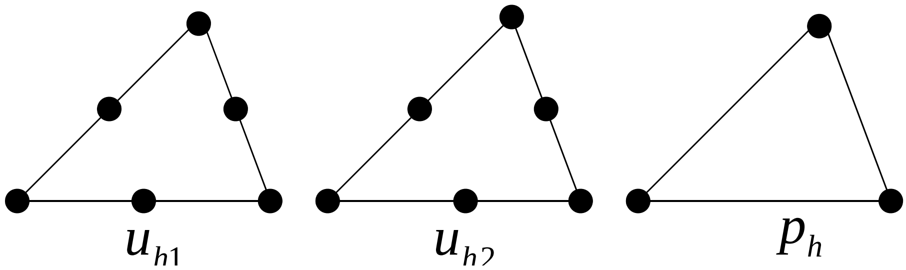
P2/P1 finite element spaces

$\mathcal{T}_h$: mesh

$\Omega_h \equiv \mathrm{int} \bigcup \{ K; K \in \mathcal{T}_h \}$ triangles ($d = 2$)

P_2 / P_1 finite element space





Interface tracking energy-stable FE scheme

$$X_h \subset X, \qquad \Psi_h \subset L^2(\Omega), \quad V_h \times Q_h \subset V \times Q,$$

Δt : time increment, $N_T \equiv \lfloor T / \Delta t \rfloor$

Find $\left\{ \left(\chi_h^n, u_h^n, p_h^n \right) \right\}_{n=0}^{N_T} \subset X_h \times V_h \times Q_h$ such that

$$D_{\Delta t} \chi_h^n = \frac{3}{2} u_h^n \left(\chi_h^n \right) - \frac{1}{2} u_h^{n-1} \left(\chi_h^n - \Delta t u_h^n \left(\chi_h^n \right) \right) \qquad \forall s_i^n$$

$$\left(\rho_h^n D_{\Delta t} u_h^n + \frac{1}{2} u_h^{n+1} D_{\Delta t} \rho_h^n, v_h \right) + a_1(\rho_h^{n+1}, u_h^n, u_h^{n+1}, v_h) + a_0 \left(\rho_h^{n+1}, u_h^{n+1}, v_h \right)$$

$$+ b\left(v_h, p_h^{n+1} \right) + \sum_{k=1}^{m} \Delta t [\sigma_k D_{\Delta \ell} u_h^{n+1}, D_{\Delta \ell} v_h]_{\Gamma_{kh}^{n+1}}$$

$$= \left(\rho_h^{n+1} \Pi_h f^{n+1}, v_h \right) - \sum_{i=1}^{m} [\sigma_i D_{\Delta \ell} \chi_h^{n+1}, D_{\Delta \ell} v_h]_{\Gamma_{ih}^{n+1}} \qquad \forall v_h \in V_h$$

$$b\left(u_h^{n+1}, q_h \right) = 0 \qquad \forall q_h \in Q_h$$

IC: $\chi_h^0 = \Pi_h \chi^0$, $u_h^0 = \Pi_h u^0$

Other issues on the computation

At each time level n, we

- control $N^n_{x,k}$, the number of particles in $\chi^n_{h,k}$ for $i = k, \cdots, m$
- adjust the positions of particles in $\chi^n_{h,k}$ to keep the area of Ω^n_{kh} identical

Algorithm of the energy-stable FE scheme

$$\left(\chi_h^n, u_h^n, p_h^n\right) \to \left(\chi_h^{n+1}, u_h^{n+1}, p_h^{n+1}\right)$$

(1) $\tilde{\chi}_h^{n+1} = \chi_h^n + \Delta t\left(\frac{3}{2}u_h^n\left(\chi_h^n\right) - \frac{1}{2}u_h^{n-1}\left(\chi_h^n - \Delta t u_h^n\left(\chi_h^n\right)\right)\right)$

(2) $\tilde{\chi}_h^{n+1} \to \chi_h^{n+1}, \qquad \rho_h^{n+1} = \rho_h^{n+1}\left(\chi_h^{n+1}\right)$

(3) Find $\left(u_h^{n+1}, p_h^{n+1}\right)$:

$$\begin{pmatrix} \frac{1}{\Delta t}M\left(\rho_h^n\right) + \frac{1}{2\Delta t}M\left(\rho_h^{n+1} - \rho_h^n\right) + A_1(\rho_h^{n+1}, u_h^n) + A_0(\rho_h^{n+1}) + \Delta t S(\chi_h^{n+1}) & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u_h^{n+1} \\ p_h^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\Delta t}M\left(\rho_h^n\right)u_h^n + M\left(\rho_h^{n+1}\right) f_h^{n+1} - S(\chi_h^{n+1})\chi_h^{n+1} \\ 0 \end{pmatrix}$$

solved by GMRES

Mass and energy stability

Suppose χ_h : "smooth", uniformly bounded length

$\Rightarrow$

$$\operatorname{meas}\Omega_{kh}^{n}=\operatorname{meas}\Omega_{kh}^{0}\quad\left(i=0,\ldots,m\right)$$

$$\left\|\sqrt{\rho_h}u_h\right\|_{\ell^\infty\left(L^2(\Omega)\right)},\ \left\|\sqrt{\mu_h}D\left(u_h\right)\right\|_{\ell^2\left(L^2(\Omega)\right)}$$

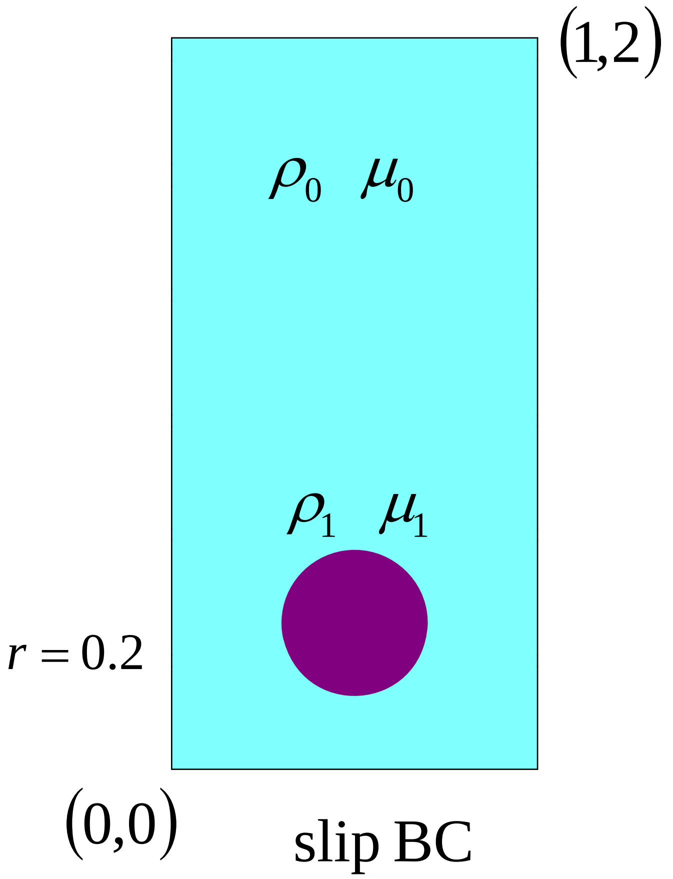
$$\le c\left(\left\|\sqrt{\rho_h}f_h\right\|_{\ell^2\left(L^2(\Omega)\right)}+\sum_{k=1}^{m}\left\|\sqrt{\sigma_k}D_{\Delta\ell}^2\chi_h\right\|_{\ell^2\left(L^2\left(\Gamma_{kh}\right)\right)}\right)$$

Note. The result is valid also in the case $\sigma_i=0$.

Note. $\left\|D_{\Delta\ell}^2\chi_h\right\|_{\ell^\infty\left(L^2\left(\Gamma_{kh}\right)\right)},\ \left\|D_{\Delta\ell}^2\chi_h\right\|_{\ell^\infty\left(L^\infty\left(\Gamma_{kh}\right)\right)}$ may be large.

Numerical Results of Multi-Phase Flow Problems

Rising bubble problem(1)



$$\rho_0 = 100 \qquad \mu_0 = 0.5,\ 5$$

$$\rho_1 = 0.1 \qquad \mu_1 = 1$$

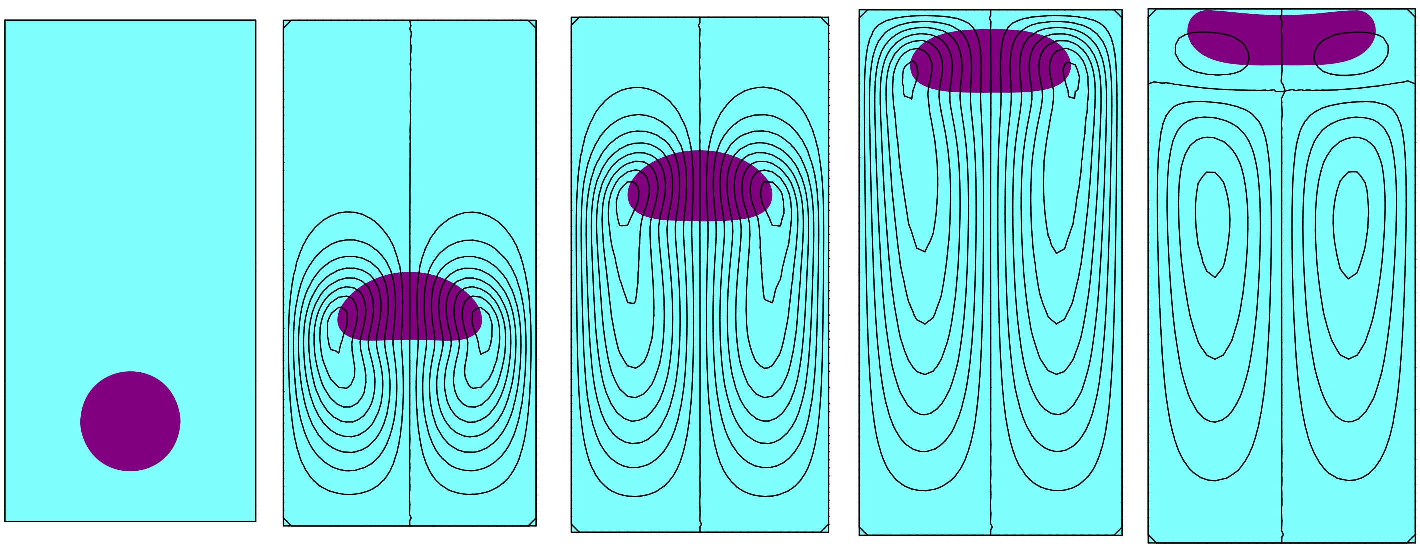
$$f = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\sigma_1 = 2.0$$

$$T = 10,\ 15$$

BB3a_e-1_5e-2

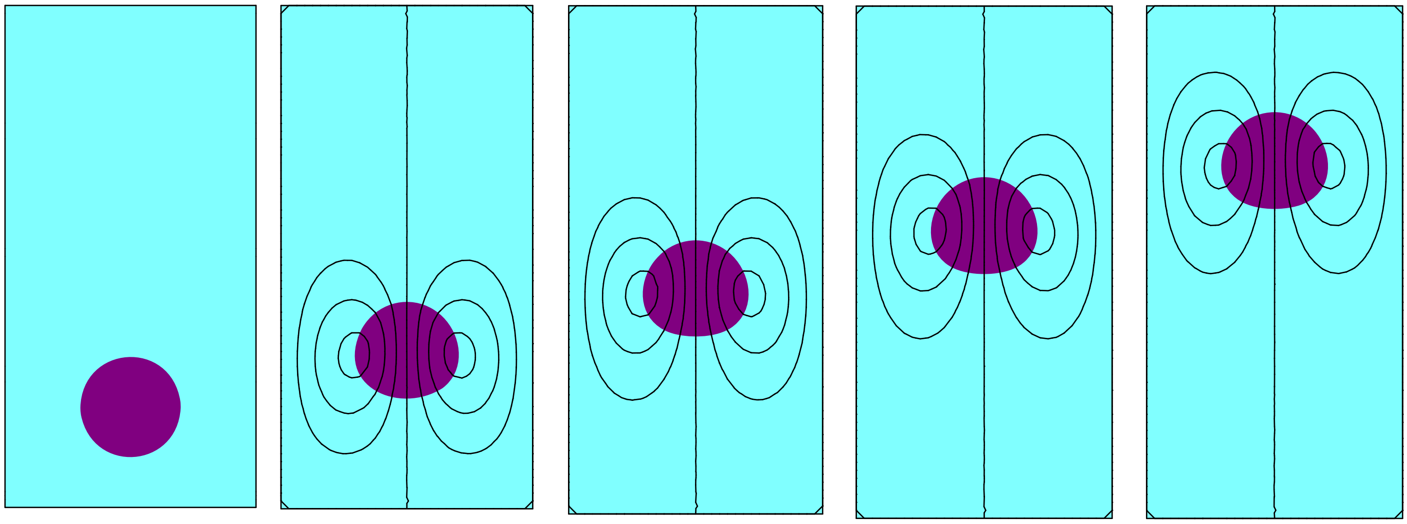
Rising bubble problem(1) $\mu_0 = 0.5$



$t = 0.0$ $\quad$ $t = 2.0$ $\quad$ $t = 4.0$ $\quad$ $t = 6.0$ $\quad$ $t = 8.0$

$$\left\| \chi_h \right\|_{l^2\left(H_0^2\right)} = 31.87$$

Rising bubble problem(1) $\mu_0 = 5.0$

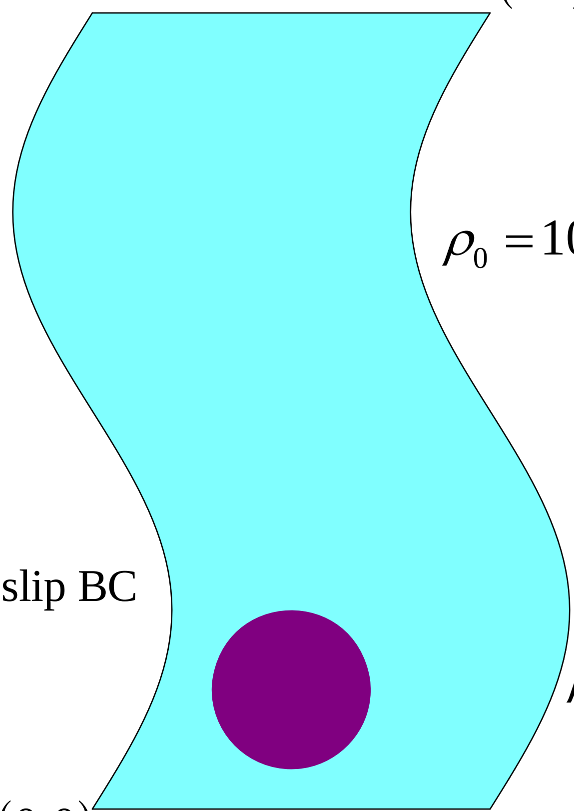


$t = 0.0$ $t = 2.0$ $t = 4.0$ $t = 6.0$ $t = 8.0$

$$\left\| \chi_h \right\|_{l^2\left(H_0^2\right)} = 27.33 \ \left(T = 15\right)$$

Rising bubble problem in a swayed channel

$(1,2)$



slip BC

$(0,0)$

$\rho_0 = 100, \mu_0 = 2$

$\rho_1 = 0.1, \mu_1 = 1$

$$f = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

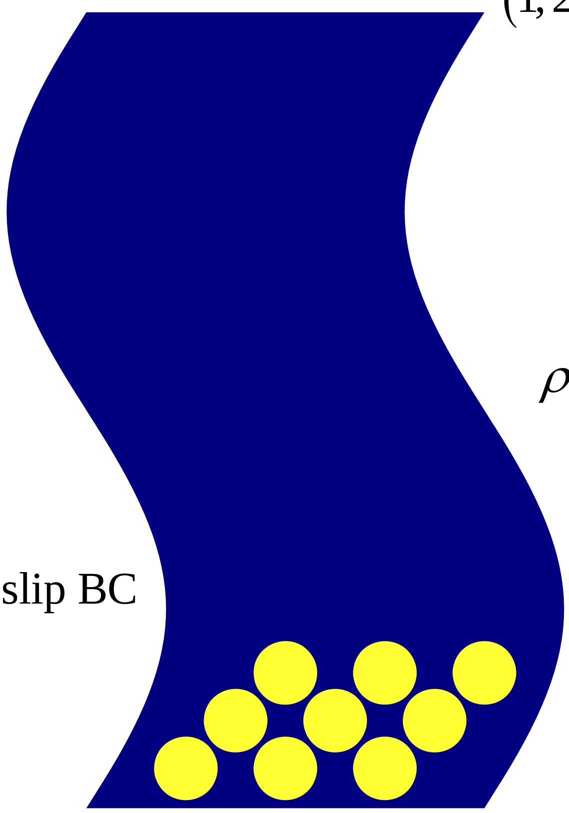
$\sigma_1 = 2$

$T = 15$

$N = 32, \quad \Delta t = \dfrac{1}{16}$

SC01s

Rising bubbles problem



$$f = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\sigma = 1$$

$$T = 30$$

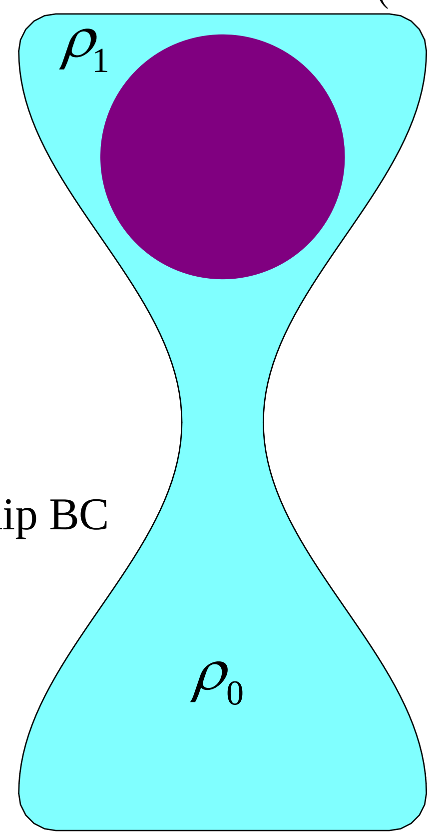
$$\rho_0 = 100, \mu_0 = 2$$

$$N = 32, \quad \Delta t = \frac{1}{64}$$

$$\rho_i = 0.1, \mu_i = 1$$
$$(i = 1,..,9)$$

NSC01s

Movement in an “Hourglass”



$$\rho_0 = 1, \quad \mu_0 = 1$$
$$\rho_1 = 100, \ \mu_1 = 2$$

$$f = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\sigma = 0.1 \qquad T = 300$$

$$N = 32, \qquad \Delta t = \frac{1}{4}$$

HG02d, 30sec

Concluding Remarks

- We have presented an "energy-stable" finite element scheme for multiphase flow problems with merger.
- We have exhibited several numerical simulations to show the robustness and applicability of the scheme.
- We have shown numerical convergence for a test problem.

◆ Future work:

convergence proof; merge/split algorithm;

high-Reynolds number problems;

2.5D problems; 3D problems; other applications

References

- M. Tabata. Finite element schemes based on energy-stable approximation for two-fluid flow problems with surface tension. Hokkaido Mathematical Journal, 36(2007), 875-890.
- M.Tabata. Numerical simulation of fluid movement in an hourglass by an energy-stable finite element scheme. In M. N. Hafez, K. Oshima, and D. Kwak, editors, Computational Fluid Dynamics Review 2010, pp. 29-50. World Scientific, Singapore, 2010.