第１１回早稲田大学「流体数学セミナー」

Navier-Stokes 方程式―ミレニアム問題の現在

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1. Navier-Stokes equations

$\mathbb{R}^3$: 3-D Euclidean space, $x = (x_1, x_2, x_3)$, $t \geq 0$: time

$$\begin{aligned} u &= u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \quad \text{velocity vector}, \\ p &= p(x,t) \quad \text{pressure} \end{aligned}$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \cdot \nabla = \frac{\partial}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} \quad \text{Lagrange differentiation}$$

(N-S)

$$\begin{cases} \dfrac{Du}{Dt} = \nu \Delta u - \dfrac{1}{\rho} \nabla p, \quad x \in \mathbb{R}^3, t > 0 \quad \text{(momentum conservation)} \\ \text{div } u = 0, \quad x \in \mathbb{R}^3, t > 0. \quad \text{(mass conservation)} \end{cases}$$

$$\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}, \quad \nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}), \quad \text{div } u = \nabla \cdot u = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j}$$

ν:kinematic viscosity, ρ:density, Assume that $\nu = \rho = 1$.

(1) $\quad u(x,0) = a(x) = (a_1(x), a_2(x), a_3(x))$ (initial data)

Cauchy Problem. For *any* given a find a pair $\{u, p\}$ of functions satisfying (N-S) for $t > 0$ with (1) at $t = 0$.

(i) (**existence of local solutions**) For $a = a(x)$, dose (N-S) have a solution $\{u(x,t), p(x,t)\}$ on $(x,t) \in \mathbb{R}^3 \times [0,T)$ for some $T > 0$?

(ii) (**uniqueness & regularity of solutions**) Is the solution unique? Is the solution infinitely many times differentiable with respect to $(x,t) \in \mathbb{R}^3 \times [0,T)$?

(iii) (**continuity of solutions for initial data**) Suppose that $\{v, q\}$ is another solution of (N-S) for the initial data $b(x)$. If $a \approx b$ on $\mathbb{R}^3$, then $\{u, p\} \approx \{v, q\}$ on $\mathbb{R}^3 \times [0,T)$?

(iv) (**global solution**) In (i), (ii) and (iii) can one take $T = \infty$?

If (i), (ii) and (iii) are affirmative for $\exists T < \infty$, then we say that the Cauchy problem to (N-S) is **locally well-posed**.

If (i), (ii) and (iii) are affirmative for $T = \infty$, then we say that the Cauchy problem to (N-S) is **globally well-posed**.

Millennium Prize Problem by Clay Math. Inst. 2000

Is (N-S) globally well-posed ?

If Yes !

$\Longrightarrow$ You will get

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Solutions to linear PDE

1. *Poisson equation*

$$-\Delta v = f, \quad x \in \mathbb{R}^3, \quad G(x) \equiv \frac{1}{4\pi}|x|^{-1}$$

$\Longrightarrow$

$$v(x) = \int_{\mathbb{R}^3} G(x-y)f(y)dy = \iiint_{\mathbb{R}^3} G(x-y)f(y)dy_1 dy_2 dy_3$$

gives a solution formula.

2. *Cauchy problem to the heat equation*

$$\frac{\partial v}{\partial t} - \Delta v = f, \quad x \in \mathbb{R}^3, t > 0, \quad v(x,0) = b(x)$$

$\Longrightarrow$

$$v(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t)b(y)dy + \int_0^t \int_{\mathbb{R}^3} \Gamma(x-y,t-\tau)f(y,\tau)dyd\tau$$

gives a solution formula, where $\Gamma(x,t) \equiv (4\pi t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4t}}$

Solution to **nonlinear** **PDE** $\Longrightarrow$ No solution formula!

Method 1; Linear perturbation

(N-S) $\approx$ perturbation from the linear Stokes equation

$$\text{(N-S')} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \nabla p = -u\cdot\nabla u, \quad x\in\mathbb{R}^3, t>0, \\ \text{div } u = 0 \quad x\in\mathbb{R}^3, t>0, \\ u(x,0) = a(x) \end{cases}$$

$\Longleftrightarrow$ (Duhamel principle)

(IE)

$$u(x,t) = \int_{\mathbb{R}^3}\Gamma(x-y,t)a(y)dy - \int_0^t\int_{\mathbb{R}^3}E(x-y,t-\tau)u\cdot\nabla u(y,\tau)dyd\tau,$$

$$E_{ij}(x,t) = \Gamma(x,t)\delta_{ij} + \frac{\partial^2}{\partial x_i\partial x_j}\int_{\mathbb{R}^3}G(x-y)\Gamma(y,t)dy, \quad i,j=1,2,3.$$

successive approximation(iteration method)

$$u^{(0)}(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t)a(y)dy,$$

$$u^{(j+1)}(x,t) = u^{(0)}(x,t) - \int_0^t \int_{\mathbb{R}^3} E(x-y,t-\tau)u^{(j)} \cdot \nabla u^{(j)}(y,\tau)dyd\tau$$

$$(j = 1, 2, \cdots)$$

existence of solution $\iff$ $u(x,t) = \exists \lim_{j\to\infty} u^{(j)}(x,t)$

In general, only **local** solution can be constructed;

$\exists T_* < \infty$ such that $\exists \lim_{j\to\infty} u^{(j)}(x,t)$ for $0 \le t < T_*$

Method 2; Variational principle

Energy conservation

$$(2) \qquad \frac{1}{2}\int_{\mathbb{R}^3}\sum_{i=1}^{3}|u_i(x,t)|^2dx + \int_0^t\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\left|\frac{\partial u_i}{\partial x_j}(x,\tau)\right|^2 dxd\tau$$

$$= \frac{1}{2}\int_{\mathbb{R}^3}\sum_{i=1}^{3}|a_i(x)|^2dx$$

for all $0 \le t < \infty$. (2) is called the **energy equality** of (N-S)-(1).

(2) $\Longrightarrow$ $\exists$ weak solution u such that

$$\max_{0<t<\infty}\int_{\mathbb{R}^3}\sum_{i=1}^{3}|u_i(x,t)|^2dx+\int_0^\infty\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\left|\frac{\partial u_i}{\partial x_j}(x,\tau)\right|^2 dxd\tau \le \int_{\mathbb{R}^3}\sum_{i=1}^{3}|a_i(x)|^2dx$$

advantage: $\exists u(\cdot,t)$ solution for all $0 < t < \infty$ (global solution)

disadvantage: smoothness of u is unknown!

Question: Can we control

$$(3)\quad \int_0^t \int_{\mathbb{R}^3} \sum_{i=1}^{3} |\Delta u_i(x,\tau)|^2 dx d\tau, \qquad \max_{t>0} \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \left| \frac{\partial u_i}{\partial x_j}(x,t) \right|^2 dx$$

by means of the initial data a ?

2. Existence of global weak solution

$$L^2_\sigma \;=\; \{u=(u_1,u_2,u_3);\, \text{div } u=0, \int_{\mathbb{R}^3}\sum_{i=1}^{3}|u_i(x)|^2dx<\infty\},$$

$$H^1_\sigma \;=\; \{u=(u_1,u_2,u_3)\in L^2_\sigma; \int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\left|\frac{\partial u_i}{\partial x_j}(x)\right|^2 dx<\infty\}$$

$$u,v\in L^2_\sigma \Longrightarrow (u,v)\equiv \int_{\mathbb{R}^3}\sum_{i=1}^{3}u_i(x)v_i(x)dx$$

$$u,v\in H^1_\sigma \Longrightarrow (u,v)_{H^1}\equiv (u,v)+(\nabla u,\nabla v), \quad \nabla u=\left(\frac{\partial u_i}{\partial x_j}\right)_{i,j=1,2,3}$$

L^2_σ, H^1_σ: Hilbert spaces $\quad H^1_\sigma\subset L^2_\sigma$

PDE theory in functional analysis

solution $u(x,t)\Longleftrightarrow$ one parameter family of t with its value in L^2_σ and H^1_σ, i.e.,

X: Hilbert space(Banach space), u: $t \in [0,T) \mapsto u(\cdot,t) \in X$,

ODE$\Longrightarrow$ $X = \mathbb{R}^1, \mathbb{R}^3, \cdots$, **finite dimensional** vector space

PDE$\Longrightarrow$ $X = L^2, H^1, \cdots$, **infinite dimensional** function space

$\| \cdot \|_X$: the norm of X,

$$L^s(0,T;X) \equiv \{u : t \in (0,T) \mapsto u(t) \in X; \int_0^T \|u(t)\|_X^s dt < \infty\}, \quad 1 \leq s < \infty$$

$$L^\infty(0,T;X) \equiv \{u : t \in (0,T) \mapsto u(t) \in X; \sup_{t \in (0,T)} \|u(t)\|_X < \infty\}$$

$$C^m([0,T);X)$$

$$\equiv \{u : t \in [0,T) \mapsto u(t) \in X, m\text{-times continuously differentiable}; \sup_{t \in [0,T)} \|\frac{d^m}{dt^m} u(t)\|_X < \infty\}$$

Definition 2.1. Let $a \in L^2_\sigma$. A function u is a **weak solution** of (N-S)–(1) on $(0, T)$ if

(i) $u \in L^\infty(0, T; L^2_\sigma)$, $\nabla u \in L^2(0, T; L^2)$;

(ii) The identity

$$\begin{aligned} &\int_0^T \{-(u(t), \frac{\partial \Phi}{\partial t}(t)) + (\nabla u(t), \nabla \Phi(t)) + (u \cdot \nabla u(t), \Phi(t))\} dt \\ = \;& (a, \Phi(0)) \end{aligned}$$

holds for all $\Phi \in C^1([0, T]; H^1_\sigma)$ with $\Phi(\cdot, T) = 0$.

(u satisfies (N-S) in the sense of **distribution**.)

Theorem 2.1. (Leray) For arbitrary $a \in L^2_\sigma$ there exists a weak solution u of (N-S)–(1) on $(0, \infty)$ such that

$$\frac{1}{2}\|u(t)\|^2_{L^2} + \int_s^t \|\nabla u(\tau)\|^2_{L^2} d\tau \le \frac{1}{2}\|u(s)\|^2_{L^2} \tag{4}$$

for a.e. $s \ge 0$, including $s = 0$, and $\forall t$ such that $0 \le s \le t < \infty$.

$$\|u(t) - a\|_{L^2} \to 0, \quad \text{as } t \to +0, \tag{5}$$

where $\|u\|_{L^2} = \sqrt{(u, u)}$.

We solved Problem (i) for $T = \infty$ by introducing the notion of **weak solutions**.

Problem (ii) Is the weak solution $u(x, t)$ in Theorem 2.1 unique ? Is $u(x, t)$ differentiable with respect to for (x, t) ?

partial answer: (4) guarantees smoothness of u to some extent.

Theorem 2.2. (Leray's structure theorem)

Suppose that u is a weak solution of (N-S)–(1) on $(0, \infty)$ with the energy inequality (4):

(S.E.I) $$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_s^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \frac{1}{2}\|u(s)\|_{L^2}^2$$

for a.e. $s \geq 0$, including $s = 0$, and $\forall t$ such that $0 \leq s \leq t < \infty$. Then $\exists \{I_k\}_{k=0}^{\infty}$: a disjoint family of intervals on $(0, \infty)$ s.t.

(i) $\exists T_0 > 0$ such that $I_0 = [T_0, \infty)$;

(ii) $|(0, \infty) \setminus \cup_{k=0}^{\infty} I_k| = 0$ and $\displaystyle\sum_{k=1}^{\infty} |I_k|^{\frac{1}{2}} < \infty$;

(iii) $u(\cdot, t) \in C^{\infty}(\mathbb{R}^3)$ for all $t \in I_k$, $(k = 0, 1, \cdots)$,

where $|I|$ denotes the length of the interval I.

Size of singular set in the space-time $\mathbb{R}^3 \times (0,T)$

For a weak solution u we denote by $S(u)$ the singular set defined by

$$S(u) \equiv \{(x,t) \in \mathbb{R}^3 \times (0,T); \sup_{(y,s)\in B_\rho(x,t)} |u(y,s)| = \infty \quad \text{for } \forall \rho > 0\},$$

where $B_\rho(x,t) = \{(y,s) \in \mathbb{R}^3 \times (0,T); |y-x| < \rho, |s-t| < \rho\}$.

Theorem 2.3.(Caffarelli-Kohn-Nirenberg) $\forall$ weak solution u with the *localized* energy inequality

$$\text{(L.E.I.)} \qquad 2\iint_{\mathbb{R}^3\times(0,T)} |\nabla u|^2 \phi dxdt$$
$$\leq \iint_{\mathbb{R}^3\times(0,T)} [|u|^2(\partial_t\phi + \Delta\phi) + (|u|^2 + 2p)u \cdot \nabla\phi] dxdt$$

for all $\phi \in C_0^\infty(\mathbb{R}^3 \times (0,T))$ with $\phi \geq 0$.

$$\Longrightarrow \qquad \mathcal{H}^1(S(u)) = 0,$$

where $\mathcal{H}^1(S)$ denotes the one-dimensional Hausdorff measure of the set S in the space-time $\mathbb{R}^3 \times (0,\infty)$.

Uniqueness and regularity of weak solutions

Theorem 2.4.(Serrin, von Wahl, Giga, Masuda, Sohr–K., Hishida-Izumida, Neustupa, Eskauriaza-Seregin-Šverák) Let $a \in L^2_\sigma$. Let u and v be two weak solutions of (N-S)–(1) on $(0,T)$. Suppose that v satisfies the energy inequality (4) with $s = 0$, i.e.,

$$\frac{1}{2}\|v(t)\|^2_{L^2} + \int_0^t \|\nabla v(\tau)\|^2_{L^2} d\tau \le \frac{1}{2}\|a\|^2_{L^2}, \quad 0 \le t < T.$$

Assume that u satisfies

$$(6) \qquad u \in L^s(0,T;L^r) \quad \text{for } 2/s + 3/r = 1 \text{ with } 3 \leqq r \leqq \infty,$$

i.e.,

$$\int_0^T \|u(t)\|^s_r dt < \infty \quad \text{for } 2/s + 3/r = 1 \text{ with } 3 \leqq r \leqq \infty.$$

Then we have $u = v$ on $\mathbb{R}^3 \times (0,T)$, and it holds

$$\frac{\partial u}{\partial t}, \nabla u, \nabla^2 u, \cdots, \nabla^k u, \cdots \in C(\mathbb{R}^3 \times (0,T)).$$

Remark. Eskauriaza-Seregin-Šverák showed by contradiction argument the crtical case $s = \infty$ and $r = 3$:

$$u \in L^\infty(0, T; L^3) \quad \Longrightarrow \quad u(t) \in C^\infty(\mathbb{R}^3), 0 < \forall t < T.$$

Problem. Direct proof of regularity result on weak solution in the class $L^\infty(0, T; L^3)$

Scaling invariance: $\lambda > 0$:parameter, a family $\{u_\lambda, p_\lambda\}$ of functions

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

$\{u, p\}$ is a solution of (N-S) on $\mathbb{R}^3 \times (0, \infty)$.

$\Longleftrightarrow$

$\{u_\lambda, p_\lambda\}_{\lambda>0}$ is a solution of (N-S) on $\mathbb{R}^3 \times (0, \infty)$.

It is easy to check that

$$
\begin{aligned}
\|u_\lambda\|_{L^s(0,\infty;L^r)} &= \left(\int_0^\infty \left(\int_{\mathbb{R}^3} |u_\lambda(x,t)|^r dx\right)^{\frac{s}{r}} dt\right)^{\frac{1}{s}} \\
&= \lambda^{1-(\frac{2}{s}+\frac{3}{r})} \left(\int_0^\infty \left(\int_{\mathbb{R}^3} |u_\lambda(x,t)|^r dx\right)^{\frac{s}{r}} dt\right)^{\frac{1}{s}} \\
&= \lambda^{1-(\frac{2}{s}+\frac{3}{r})} \|u\|_{L^s(0,\infty;L^r)}
\end{aligned}
$$

holds for all $\lambda > 0$. This implies that the space (6)

$$L^s(0,\infty;L^r) \quad \text{for } 2/s + 3/r = 1 \text{ with } 3 \leqq r \leqq \infty$$

is *invariant* under the change of scale such as $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$.

Importance!(*Fujita-Kato principle*) Find a solution u in a function space Y on $\mathbb{R}^3 \times (0,\infty)$ such as $\|u_\lambda\|_Y = \|u\|_Y$ holds for all $\lambda > 0$.

Further results. Larger spaces for regularity of weak solutions

Let $\phi = \phi(\xi) \in C_0^\infty(\mathbb{R}^3)$ be as

$$\text{supp } \phi \subset \{\xi \in \mathbb{R}^3; 1/2 \le |\xi| \le 2\}, \quad \phi(\xi) > 0 \quad \text{for } 1/2 < |\xi| < 2,$$

$$(7) \sum_{k=-\infty}^{\infty} \phi(2^{-k}\xi) = 1 \quad \text{for } \xi \neq 0$$

Define $\{\varphi_k\}_{k\in\mathbb{Z}}$(Littlewood-Paley functions) so that

$$\varphi_k(\xi) \equiv \mathcal{F}^{-1}\phi(2^{-k}\cdot) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} e^{ix\xi}\phi(2^{-k}\xi)d\xi, \quad k = 0, \pm 1, \cdots$$

By (7) $f \in \mathcal{S}$ can be expressed by

$$f = \sum_{k=-\infty}^{\infty} \varphi_k * f \quad (\textit{Littlewood-Paley decomposition} \text{ of } f).$$

Defintion.(Besov & Triebel-Lizorkin spaces $\dot{B}^s_{p,q}$, $\dot{F}^s_{p,q}$)

$$\dot{B}^s_{p,q} = \{f \in \mathcal{S}'/\mathcal{P}; \|f\|_{\dot{B}^s_{p,q}} \equiv \left(\sum_{k=-\infty}^{\infty} \left(2^{ks} \|\varphi_k * f\|_{L^p} \right)^q \right)^{\frac{1}{q}} < \infty\},$$

$$s \in \mathbb{R}, 1 \leqq p \leqq \infty, 1 \leqq q < \infty,$$

$$\dot{B}^s_{p,\infty} = \{f \in \mathcal{S}'/\mathcal{P}; \|f\|_{\dot{B}^s_{p,\infty}} \equiv \sup_{k \in \mathbb{Z}} \left(2^{ks} \|\varphi_k * f\|_{L^p} \right) < \infty\},$$

$$s \in \mathbb{R}, 1 \leqq p \leqq \infty,$$

$$\dot{F}^s_{p,q} = \{f \in \mathcal{S}'/\mathcal{P}; \|f\|_{\dot{F}^s_{p,q}} \equiv \left(\int_{\mathbb{R}^3} \left(\sum_{k=-\infty}^{\infty} \left(2^{ks} |\varphi_k * f(x)| \right)^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty\},$$

$$s \in \mathbb{R}, 1 \leqq p < \infty, 1 \leqq q \leqq \infty$$

$$\dot{F}^s_{\infty,q} = \{f \in \mathcal{S}'/\mathcal{P};$$

$$\|f\|_{\dot{F}^s_{\infty,q}} \equiv \sup_{Q:\text{dyadic}} \left(\frac{1}{|Q|} \int_Q \sum_{k=-\log_2 l(Q)}^{\infty} (2^{ks} |\varphi_k * f(x)|)^q dx \right)^{\frac{1}{q}} < \infty\},$$

$$s \in \mathbb{R}, 1 \leqq q \leqq \infty$$

Proposition 2.5. (i) $s \in \mathbb{R}$, $1 \leqq p \leqq \infty$, $1 \leqq q_1 \leqq q_2 \leqq \infty$
$\Longrightarrow$

$$\dot{B}^s_{p,q_1} \subset \dot{B}^s_{p,q_2}, \quad \dot{F}^s_{p,q_1} \subset \dot{F}^s_{p,q_2}$$

(ii) $s \in \mathbb{R}$, $1 < p < \infty$
$\Longrightarrow$

$$\begin{aligned}
&\dot{B}^s_{p,q} \subset \dot{F}^s_{p,q} \subset \dot{B}^s_{p,p}, \quad 1 < q \leqq p < \infty \\
&\dot{B}^s_{p,p} \subset \dot{F}^s_{p,q} \subset \dot{B}^s_{p,q}, \quad 1 < p \leqq q < \infty \\
&\dot{B}^s_{p,p} = \dot{F}^s_{p,p}.
\end{aligned}$$

(iii) $s > 0$, $1 < p < \infty$
$\Longrightarrow$

$$\dot{F}^s_{p,2} = \dot{H}^s_p \equiv \{f \in \mathcal{S}'; \|f\|_{\dot{H}^s_p} \equiv \|(-\Delta)^{\frac{s}{2}} f\|_{L^p} < \infty\}.$$

(iv) $s = 0$, $p = 1, \infty \Longrightarrow$

$$\begin{aligned}
\dot{F}^0_{1,2} &= \mathcal{H}^1 \quad \text{Hardy space} \\
&= \{f \in L^1; Mf(x) = \sup_{t>0} |\psi_t * f(x)| \in L^1\}, \quad \psi_t(x) = t^{-n}\psi(x/t)
\end{aligned}$$

$$\begin{aligned}\dot{F}^0_{\infty,2} &= BMO \quad \text{bounded mean oscillation} \\ &= \{f \in L^1_{loc}; \|f\|_{BMO} \equiv \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f(x) - f_B|dx < \infty\}\end{aligned}$$

(v) $(\mathcal{H}^1)^* = BMO$

Coifman-Lions-Meyer-Semmes

$$u \in W^{1,2} \Longrightarrow u \cdot \nabla u \in \mathcal{H}^1$$

Theorem 2.7.(Taniuchi-K, Shimada-K) *Let $a \in L^2_\sigma(\mathbb{R}^n)$. Let u be a weak solution on (N-S)-(1). If*

$$u \in L^2(0,T; BMO), \quad \textit{or}$$

(8) $$u \in L^s(0,T; \dot{F}^{-\alpha}_{\infty,\infty}) \quad \textit{for } 2/s = 1-\alpha \textit{ with } 0 \leqq \alpha < 1$$

Then it holds that $u \in C^\infty(\mathbb{R}^3 \times (0,T))$

Remark. $L^r \subset \dot{F}^{-3/r}_{\infty,\infty}$, $3 < r \leqq \infty$. Hence (8) covers the Serrin class (6) except for $r = 3$.

Theorem 2.8.(Farwig-Sohr-Varnhorn) *Let* $a \in L^2_\sigma$. *The weak solution* u *of (N-S)-(1) on* $(0, T)$ *satisfies*

$$u \in L^s(0, T; L^r) \quad \textit{for } 2/s + 3/r = 1 \textit{ with } 3 < r < \infty$$

if and only if $a \in \dot{B}^{-1+3/r}_{r,s}$.

Local properties of weak solutions.

Removable singularity for 3-D harmonoic functions :

Let $u \in C^2(B_\delta(x_0) \setminus \{x_0\})$ and $\Delta u = 0$ in $B_\delta(x_0) \setminus \{x_0\}$, where $B_\delta(x_0) = \{x \in \mathbb{R}^3; |x - x_0| < \delta\}$.

If

$$u(x) = o(|x - x_0|^{-1}) \quad \text{as } x \to x_0,$$

then there exists $\tilde{u} \in C^2(B_\delta(x_0))$ with $\Delta \tilde{u} = 0$ in $B_\delta(x_0)$ such that $\tilde{u}(x) = u(x)$ for $x \in B_\delta(x_0) \setminus \{x_0\}$.

Definition. Let u be a weak solution of (N-S) on $\mathbb{R}^3 \times (0, T)$. $(x_0, t_0) \in \mathbb{R}^3 \times (0, T)$: a *regular* point

$\Longleftrightarrow$

$\exists \delta > 0$, $\exists \sigma > 0$ s. t.

$$u \in C^{2,1}(B_\delta(x_0) \times (t_0 - \sigma, t_0 + \sigma)).$$

Theorem 2.9. (K., Kim-K.) $\exists \varepsilon_0 > 0$ *s.t if a weak solution* u *satisfies at* $(x_0, t_0) \in \mathbb{R}^3 \times (0, T)$

$$\sup_{t_0 - \sigma < t < t_0 + \sigma} \|u(t)\|_{L^3_{\mathrm{w}}(B_\delta(x_0))} \le \varepsilon_0 \tag{9}$$

for $\exists \delta > 0$, $\exists \sigma > 0$

$\Longleftrightarrow$

(x_0, t_0) *is a regular point.*

Here $\|\cdot\|_{L^3_{\mathrm{W}}(B_\delta(x_0))}$ *denotes the weak* L^3*-norm, i.e.,*

$\|u\|_{L^3_{\mathrm{W}}(B_\delta(x_0))} = \sup_{R>0} R\mu\{x \in B_\delta(x_0); |u(x)| > R\}^{\frac{1}{3}}$ *(μ; Lebesgue measure).*

Example.

$$u(x) = \varepsilon_0 |x - x_0|^{-1} \quad \Longrightarrow \quad \int_{B_\delta(x_0)} |u(x)|^3 dx = \infty \quad \text{for all } \delta > 0.$$

However, we have

$$\|u\|_{L^3_{\mathrm{W}}(B_\delta(x_0))} = \frac{4}{3}\pi\varepsilon_0 \quad \text{for all } \delta > 0.$$

Notice that the weak-norm $\|\cdot\|_{L^3_{\mathrm{W}}(B_\delta(x_0))}$ cannot be small even though we take the radius δ small.

Corollary. (Removable Singularities) $\exists \varepsilon_0 > 0$ *s. t. if* (x_0, t_0) *is an isolated singular point of* u *satisfying*

$$\limsup_{x \to x_0, t \to t_0} |x - x_0||u(x,t)| < \varepsilon_0, \tag{10}$$

then (x_0, t_0) *is a regular point.*

In particular, if u *behaves at* (x_0, t_0) *like*

$$u(x,t) = o(|x - x_0|^{-1}) \quad \text{as } x \to x_0 \tag{11}$$

uniformly with respect to t *in some neighbourhood of* t_0*, then* (x_0, t_0) *is a regular point.*

3. Local existence of classical solution.

Under which initial data a can we construct the weak solution u of (N-S)-(1) with (6).

$$
\begin{aligned}
L^r &\equiv \{u = (u_1, u_2, u_3); \|u\|_{L^r} = \left(\int_{\mathbb{R}^3} |u(x)|^r dx\right)^{\frac{1}{r}} < \infty\} \\
L^r_\sigma &\equiv \{u \in L^r; \text{div } u = 0\}
\end{aligned}
$$

Theorem 3.1. (Fujita-Kato, Kato, Giga, Giga-Miyakawa) Let $3 \le r < \infty$ and let $a \in L^r_\sigma$. Then there exist $T_* > 0$ and a unique solution u of (N-S)–(1) on $(0, T_*)$ such that

$$u \in C([0, T_*); L^r_\sigma) \tag{12}$$

$$\frac{\partial u}{\partial t}, \Delta u \in C((0, T_*); L^r_\sigma) \tag{13}$$

If in addition $a \in L^r_\sigma \cap L^2_\sigma$, then u is also a weak solution of (N-S)–(0.1) on $(0, T_*)$ with the energy equality (2) for $0 \le t \le T_*$.

Remark. (i) By (12) we see that $u(t)$ is a *classical* solution on $\mathbb{R}^3 \times (0, T_*)$.

(ii) T_*: time interval of local classical solution

(14) $$T_* = \frac{C}{\|a\|_{L^r}^{\frac{2r}{r-3}}} \quad \text{for } 3 < r < \infty,$$

where $C = C(r)$ is a constant independent of a.

$$\|a\|_{L^r} \ll 1 \Longrightarrow T_* \gg 1,$$
$$1 \ll \|a\|_{L^r} \Longrightarrow T_* \ll 1$$

(iii) Question: Can we represent T_* for $a \in L^3_\sigma$?

Corollary 3.2.(**global classical solution of small data**) There is $\delta > 0$ such that if $a \in L^3_\sigma$ satisfies $\|a\|_{L^3} \leq \delta$, then we have in Theorem 3.1 that $T_* = \infty$.

Further results. (i) Local existence of strong solution for large class of initial data: Cannone, Yamazaki-K, Sawada, Ogawa-Taniuchi-K, Koch-Tataru

$$
\begin{aligned}
X &= B^{-1+\frac{3}{r}}_{r,\infty} \quad \text{for } 3 < r \leqq \infty, \\
X &= VMO^{-1} \equiv \text{closure of } C^{\infty}_{0,\sigma} \text{ in } BMO^{-1} \approx \dot{F}^{-1}_{\infty,2}
\end{aligned}
$$

$\forall a \in X$

$\Longrightarrow \exists T_* > 0$ & $\exists u$: solution of (N-S)–(1) with $u \in C_w([0, T_*); X)$

(ii) Uniqueness of strong solution for large class of initial data: Miura

$$
u \in C([0, T); F^{-1}_{\infty,2}) \cap L^{\infty}_{loc}(0, T; L^{\infty})
$$

$\Longrightarrow u$ is unique.

(iii) Ill-posedness in $\dot{B}^{-1}_{\infty,\infty}$: Bourgain-Pavlović

$\forall\delta > 0$, $\exists a \in \mathcal{S}$ with $\|a\|_{\dot{B}^{-1}_{\infty,\infty}} \leq \delta$ and $\exists u$: solution of (N-S)-(1) on $(0, \delta)$ such that

$$\|u(t)\|_{\dot{B}^{-1}_{\infty,\infty}} > 1/\delta \qquad \text{for } 0 < \exists t < \delta.$$

cf. Yoneda: Ill-posedness in $\dot{B}^{-1}_{p,\infty}$ for $2 < p \leqq \infty$

Question.

(i)(*continuation*) $u(t) \in C^\infty(\mathbb{R}^3)$ for $t \geq T_*$?

or

(ii) (*blow-up*) $\lim_{t \uparrow T_*} \|u(t)\|_{L^r} = \infty$?

Consider the vorticity rot $u \equiv \omega = (\omega_1, \omega_2, \omega_3)$, where

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \quad \omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \quad \omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

Theorem 3.3. (Ogawa-Taniuchi-K.,Yatsu-K.) Let $a \in L^r_\sigma$, $3 \le r < \infty$. Suppose that u is a solution of (N-S)–(1) on $(0, T_*)$ with (12) and (13). If

$$\int_0^{T_*} \|\omega_i(t)\|_{\dot{B}^0_{\infty,\infty}} dt < \infty, \quad i = 1, 2, 3, \tag{15}$$

or

$$\int_0^{T_*} \|\omega_i(t)\|_{BMO} dt < \infty, \quad i = 1, 2, \tag{16}$$

then there exists $T' > T_*$ such that u can be extended to the solution on $(0, T')$ of (N-S)–(1) as

$$u, \ \frac{\partial u}{\partial t}, \ \Delta u \in C(0, T'); L^r_\sigma). \tag{17}$$

Remarks. (i) Beale-Kato-Majda showed that if

$$\int_0^{T_*} \|\omega_i(t)\|_{L^\infty} dt < \infty, \quad i = 1, 2, 3, \tag{18}$$

then $\exists T' > T_*$ such that (18) holds. Notice that

$$\|\omega\|_{\dot{B}^0_{\infty,\infty}} \le C\|\omega\|_{BMO} \le C\|\omega\|_{L^\infty}, \quad \|\omega\|_{L^\infty} \equiv \sup_{x\in\mathbb{R}^3} |\omega(x)|.$$

(ii) Vortex equation in $\mathbb{R}^3$

$$\frac{\partial\omega}{\partial t} - \Delta\omega + u\cdot\nabla\omega - \omega\cdot\nabla u = 0$$

On the other hand, in $\mathbb{R}^2$ for $u = (u_1, u_2)$ we have

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} : \quad \text{scalar function}$$

with

$$\frac{\partial\omega}{\partial t} - \Delta\omega + u\cdot\nabla\omega = 0.$$

Maximum principle $\Longrightarrow$

$$\sup_{0<t<T} \|\omega(t)\|_{L^\infty(\mathbb{R}^2)} \le \|\text{rot } a\|_{L^\infty(\mathbb{R}^2)}. \quad \text{(18) is always OK.}$$

(iii) The criterion (15) holds also for the equation of perfect fluids, i.e., the Euler equations.

$$
\text{(E)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p, \quad x \in \mathbb{R}^3, t > 0 \\ \text{div } u = 0, \quad x \in \mathbb{R}^3, t > 0. \end{cases}
$$

Question. Does the criterion (16) holds also for (E) ?

5. Stability of solutions

5.1. Energy decay

Leray's problem. Let u be a weak solution of (N-S). Is it true

$$\|u(t)\|_{L^2} \to 0 \quad \text{as } t \to \infty \text{ ?}$$

Masuda :

$\forall$weak solution u with

(S.E.I.) $$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_s^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \le \frac{1}{2}\|u(s)\|_{L^2}^2$$

for a.e. $s \ge 0$, including $s = 0$, and $\forall t$ s.t. $s \le t < \infty$

$\Longrightarrow$

$$\|u(t)\|_{L^2} \to 0 \quad \text{as } t \to \infty \text{ .}$$

Wiegner:

Let $a \in L^2_\sigma$ and let u be a weak solution with (S.E.I). Suppose that

$$\|e^{t\Delta}a\|_2 = O(t^{-\alpha}) \quad \text{as } t \to \infty.$$

Then we have

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & \text{if } 0 \le \alpha \le 5/4, \\ O(t^{-\frac{5}{4}}) & \text{if } 5/4 < \alpha < \infty \end{cases}$$

as $\to \infty$.

Remark. Fujigaki-Miyakawa:

$$\forall a \in L^2_\sigma \quad \text{with} \quad \int_{\mathbb{R}^3} (1+|x|)|a(x)|dx < \infty$$

$\Longrightarrow$

$\exists u$: weak solution of (N-S)–(1) with $\|u(t)\|_{L^2} = O(t^{-\frac{5}{4}})$ as $t \to \infty$

5.2. Stability of weak solutions in Serrin's class

Assume that $a \in L^2_\sigma$ and $\bar{f} \in L^2(0,T;L^2)$ for all $T > 0$. We consider a weak solution u of (N-S) with $u|_{t=0} = a$ and with the external force f on the R.H.S:

(N-S)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = \bar{f}, \quad \text{div } u = 0 \quad x \in \mathbb{R}^3, t > 0, \\ u\,\big|_{t=0} = a, \end{cases}$$

Then the stabilty of u can be reduced to find a **global** solution v of the equation:

(N-S')

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + v \cdot \nabla v + \nabla q = \bar{f} + f, \quad \text{div } v = 0, \quad x \in \mathbb{R}^3, t > 0, \\ v\,\big|_{t=0} = a + b, \end{cases}$$

where b and f denote perturbations of initial distubance and the external force, respectively.

Theorem 5. (K.) *Let* $a \in L^2_\sigma$, $\bar{f} \in L^1(0,\infty; L^2) \cap L^\alpha(0,\infty; L^2)$ *for* $4/3 < \alpha < 2$. *Suppose that* u *is a weak solution of (N-S) in the class*

$$u \in L^s(0,\infty; L^r) \quad \text{for } 2/s + 3/r = 1 \text{ with } 3 < r \le \infty. \tag{19}$$

Assume that $b \in L^2_\sigma$, $f \in L^1(0,\infty; L^2) \cap C(0,\infty; L^2)$ *with*

$$\|f(t)\|_{L^2} = O(t^{-1}) \quad \text{as } t \to \infty.$$

Assume also that the weak solution v *of (N-S') satisfies the energy inequality of the stronger form:*

$$\frac{1}{2}\|v(t)\|_{L^2}^2 + \int_s^t \|\nabla v\|_{L^2}^2 d\tau \le \frac{1}{2}\|v(s)\|_{L^2}^2 + \int_s^t (\bar{f} + f, v) d\tau \tag{20}$$

for a.e. $s \ge 0$, *including* $s = 0$, *and* $\forall t$ *s.t.* $s \le t < \infty$. *Then* v *converges to* u *like*

$$\|v(t) - u(t)\|_{L^r} = o(t^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{r})}), \quad \|\nabla v(t) - \nabla u(t)\|_{L^r} = o(t^{-\frac{3}{2}(\frac{1}{2} - \frac{1}{r}) - \frac{1}{2}})$$

for $2 \le r < \infty$ *as* $t \to \infty$

Remark. In the above theorem, u may not be small; we need only that u belongs to the Serrin class (19). Moreover, the perturbations b and f may be large.

Exterior problems

$\exists$ Many Interesting Results

e.g, Flow past an obstacle:

$$u|_{\partial\Omega} = 0, \quad u(x,t) \to u^{\infty} \in \mathbb{R}^3 (\neq 0) \quad \text{as } |x| \to \infty$$

Finn, Masuda, Heywood, Kobayashi-Shibata, Enomoto-Shibata, Shibata-Yamazaki:

Analysis of **Oseen** operator $Lu = -P(\Delta u + u^{\infty} \cdot \nabla u)$

$\exists$ Challenging Open Problems

e.g., Flow around a rotating obstacle:

$$u|_{\partial\Omega} = \omega \times x, \quad u(x,t) \to 0 \quad \text{as } |x| \to \infty$$

(ω: angular velocity) Galdi, Hishida, Geissert-Heck-Hieber, Farwig-Hishida-Müller, Farwig-Neustupa, Hishida-Shibata