Measure valued solutions of the 2D Keller–Segel system

joint work:

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January 28, 2011

Keller-Segel system [1970]

diffusion aggregation

$$(\mathrm{KS})\quad \begin{cases} u_t = \Delta u - \mathrm{div}\left(u \nabla v\right), & x \in \Omega, \quad t>0 \\ -\Delta v = u - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u\,dx, & x \in \Omega, \quad t>0 \\ \partial_\nu u(x,t) = 0, \quad \partial_\nu v(x,t) = 0, & x \in \partial\Omega, \quad t>0 \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$$

$u\left(x,t\right)$ **density of amoebae**

$v\left(x,t\right)$ **concentration of the chemical substances**

$\Omega \subset R^2$: bounded

$\partial\Omega$: smooth

I. Known result

local classical solution

$$0 \leq u_0 \in L^1(\Omega), \qquad \Omega \subset R^2 : \text{bounded}$$

$$\exists T_0 > 0, \exists 1 (u, v) : \text{classical sol of (KS) on } [0, T_0)$$

Remark 1

- scale invariant norm: $\|u_0\|_{L^1}$
- small data global existence: $\|u_0\|_{L^1} \ll 1$

$$\exists 1 (u, v) : \text{classical sol of (KS) on } [0, \infty)$$

- large data finite time blow-up: $\|u_0\|_{L^1} \gg 1$

$$\limsup_{t \to T_*} \|u(t)\|_{L^\infty} = \infty \quad \text{for some } T_* < \infty$$

Scaling invariant transform

$$\{u, v\}: \text{ sol of (KS)}$$

$\Longleftrightarrow$

$$\{u_\lambda, v_\lambda\}: \text{ sol of (KS)} \qquad \text{for } \forall \lambda > 0$$

where

$$u_\lambda(x,t) := \lambda^2 u(\lambda x, \lambda^2 t),$$
$$v_\lambda(x,t) := v(\lambda x, \lambda^2 t)$$

$\forall N \geq 2$

$$\|u_{0,\lambda}\|_{L^{\frac{N}{2}}(R^N)} = \|u_0\|_{L^{\frac{N}{2}}(R^N)} \qquad \text{for } \forall \lambda > 0$$

$\Longleftrightarrow$

$$N = 2 \qquad \|u_{0,\lambda}\|_{L^1(R^2)} = \|u_0\|_{L^1(R^2)} \qquad \text{for } \forall \lambda > 0$$

Mass conservation law

$$\{u,v\}: \ \text{sol of (KS)} \qquad \text{on} \ \ \Omega\times[0,T)$$

$$\Rightarrow \quad \|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \qquad 0\leq \forall t<T$$

(location of blow-up points)

sharp ε – regularity thm

Nagai-Senba-Suzuki, 2001

Luckhaus-S.-Velazquez

Mass cons. law

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad 0 \le \forall t < \infty$$

Scaling invarant norm

$$\sup_{0<t<\infty} \|u(t)\|_{L^1(R^2)} = \sup_{0<t<\infty} \|u_\lambda(t)\|_{L^1(R^2)}$$

new result

$$\sup_{0<t<T} \int_{B(x_0,2\rho_0)} u(x,t)\,dx < 8\pi$$

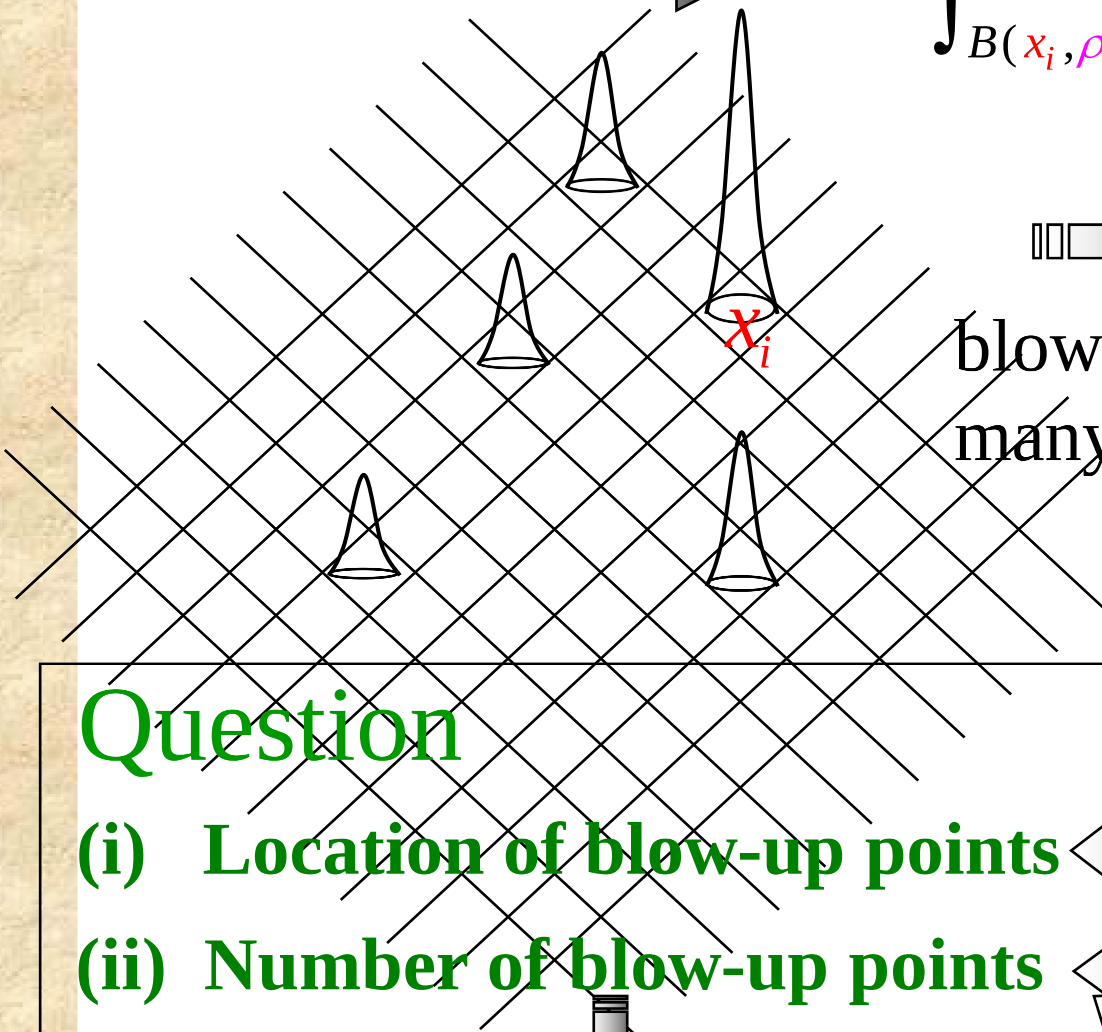
⇒ indep. of x_0

$$\sup_{(x,t)\in B(x_0,\rho_0)\times(t_0-c(\rho_0)^2,t_0)} u(x,t) < C \text{ p. of } x_0$$

$x_i \in R^N$: blow-up point of u at the time t_0

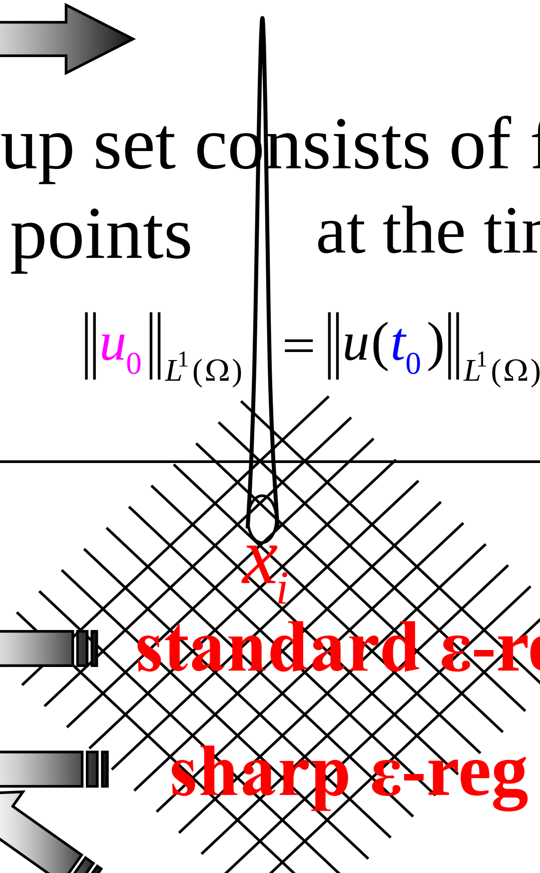
$$\int_{B(x_i,\rho)} u(x,t_0)\,dx > 8\pi$$

for $\forall \rho > 0$



blow-up set consists of finitely many points at the time t_0

$$\|u_0\|_{L^1(\Omega)} = \|u(t_0)\|_{L^1(\Omega)}$$



Question

(i) Location of blow-up points ⇐ **standard ε-reg thm**

(ii) Number of blow-up points ⇐ **sharp ε-reg thm**

(iii) δ-functional singularity ⇐ $C_w([0,T_*];L^1(\Omega))$

$$u \in C_w([0,T_*];L^1(\Omega))$$

Question

(i) Location of blow-up points

(ii) Number of blow-up points

(iii) δ-functional singularity

$\exists u^* \in L^1(\Omega)$ s.t.

$$\int_\Omega u(x,t)\varphi(x)\,dx \to \int_\Omega u^*(x)\varphi(x)\,dx$$

as $t \to T_*$

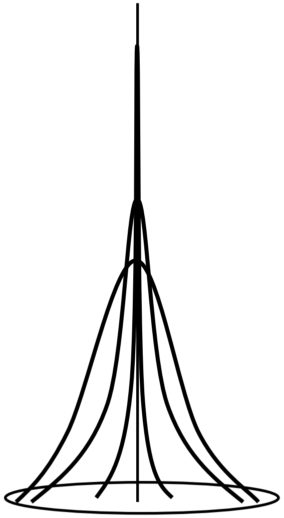
for $\forall \varphi \in L^\infty(\Omega)$

$t = T_* :$ blow-up time

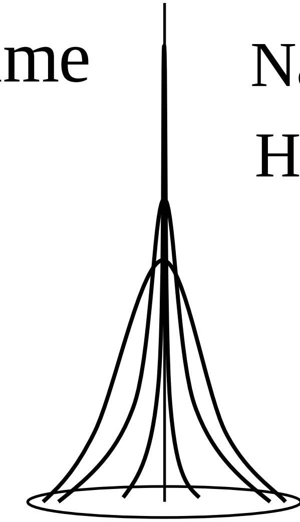
Nagai・Senba・Suzuki …

Herrero・Velazquez …

$\|u_0\|_{L^1} > 8\pi$



$\approx \alpha_1(T_*) \cdot \delta_{x_1(T_*)}$



$\approx \alpha_2(T_*) \cdot \delta_{x_2(T_*)}$

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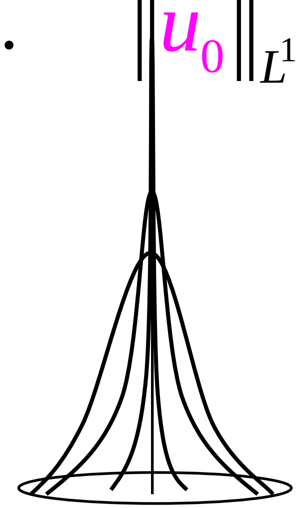


$\approx \alpha_k(T_*) \cdot \delta_{x_k(T_*)}$

k : the number of the blow-up points $\quad k \le \dfrac{\|u_0\|_{L^1(\Omega)}}{8\pi}$

$\alpha_i(T_*) \ge 8\pi, \quad i = 1, 2, \cdots, k$

Question

(i) Location of blow-up points ⇐ **standard ε-reg thm**

(ii) Number of blow-up points ⇐ **sharp ε-reg thm**

(iii) δ-functional singularity ⇐ $C_w([0, T_*]; L^1(\Omega))$

The existence of minimal immersions of 2 -spheres
J.Sacks, and K.Uhlenbeck
Ann. of Math, Vol. 271, No. 2, 1981, pp. 639-652.

Partial regularity of suitable weak solutions of the Navier-Stokes equations
L.A.Caffarelli, R.Kohn, and L.Nirenberg
Comm. Pure Appl. Math., Vol. 35, 1982, pp 771-831

II. Purpose

Construction of   Thm 1 Thm 3

"measure valued sol" of (KS) on $[0,\infty)$

for large initial data u_0 s.t. $\|u_0\|_{L^1} \gg 1$

with a simple expression of solution

such as that at T_*

Remark 1

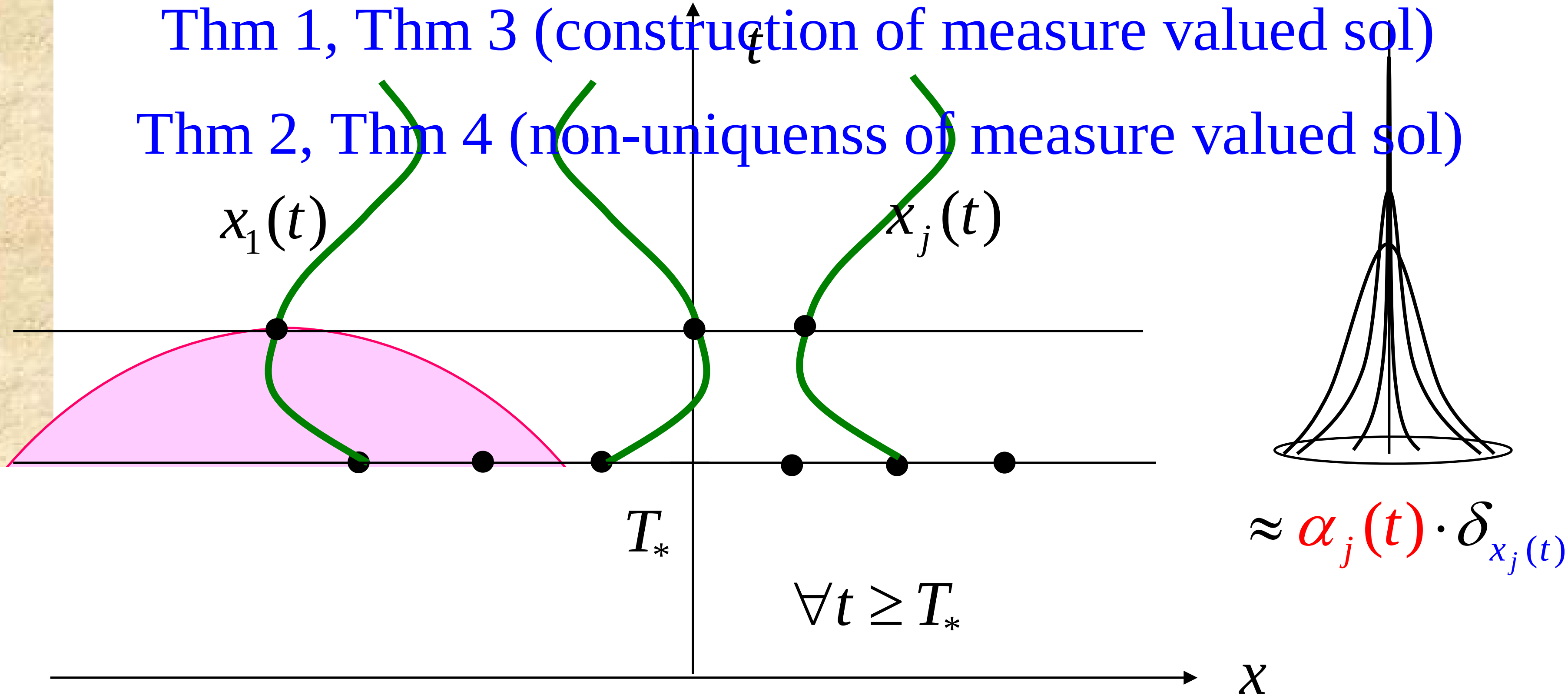
(1) strong solution is unique

(2) measure valued solution is unique ?

Thm 2, Thm 4 (non-uniquenss of measure valued sol)

Thm 1, Thm 3 (construction of measure valued sol)

Thm 2, Thm 4 (non-uniquenss of measure valued sol)



$$u^{\varepsilon_k} \to \mu, \quad d\mu = d\mu_t dt$$

(i) $\mu_t = \sum_{x_j(t)\in S_t} \alpha_j(t)\delta_{x_j(t)} + u(\cdot,t)\,dx$ a.e. $t \in [0,\infty)$

(ii) $u \in L^\infty(0,\infty; L^1(\Omega))$

(iii) $S_{t_0} \equiv S \cap \{(x,t);\ t = t_0\}$ consists of at most finitely many points

III. Our results

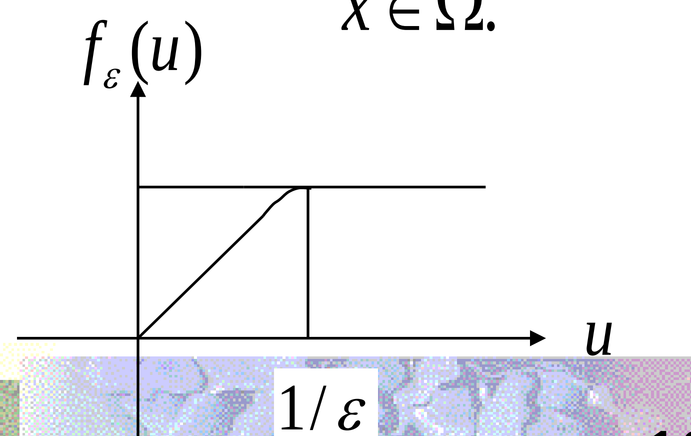
$$-\mathrm{div}\left(f_\varepsilon(u)\nabla v\right)=\begin{cases} u & u>>1 \\ u^2 & u<<1 \end{cases}$$

Regularization: I

$$(\mathrm{KS})_\varepsilon^1 \begin{cases} u_t=\Delta u-\mathrm{div}\left(f_\varepsilon(u)\nabla v\right), & x\in\Omega, \quad t>0 \\ -\Delta v=f_\varepsilon(u)-\dfrac{1}{|\Omega|}\displaystyle\int_\Omega f_\varepsilon(u)d\ , & x\in\Omega, \quad t>0 \\ \partial_\nu u(x,t)=0, \quad \partial_\nu v(x,t)=0 & x\in\partial\Omega, \quad t>0 \\ u(x,0)=u_0(x), & x\in\Omega. \end{cases}$$

$$f_\varepsilon(u)=\int_0^u \min\{1,(1/\varepsilon-s)_+\}ds$$

$\leq u$



Theorem 1 [Luckhaus-S-Velázquez.]

$f_\varepsilon(u^\varepsilon) \le u^\varepsilon$

$\exists u^\varepsilon$: sol. of $(KS)^1_\varepsilon$ on $[0,\infty)$,

$\exists$Radon measure $\mu, \mu^- \in M^+(\Omega \times [0,\infty))$ and $\exists \{\varepsilon_k\}_{k=1}^\infty$

s.t. $u^{\varepsilon_k} \to \mu, \quad f_{\varepsilon_k}(u^{\varepsilon_k}) \to \mu^-$ in the weak-* topology

and $\mu^- \le \mu$

Moreover, $d\mu = d\mu_t dt, \; d\mu^- = d\mu_t^- dt$

with $\mu_t(\Omega) = \int_\Omega u_0(x)\, dx, \quad \mu_t^- \le \mu_t$ for all $0 < t < \infty$

Moreover,

the singular set S of $\mu, \mu^- \subset \overline{\Omega} \times [T_*, \infty)$ for some $0 < T_* < \infty$

with $\mu = \overline{\mu} + u, \; \mu^- = \overline{\mu}^- + u,$

where $\overline{\mu}, \; \overline{\mu}^- \in M^+(\Omega \times [0,\infty))$

and $u \in C^\infty((\Omega \times [0,\infty)) \setminus S)$ satisfying (KS) in $\overline{\Omega} \times [0, T_*)$

Moreover, for a.e. $t_0 \in [0,\infty)$ $d\mu = d\mu_t dt, \; d\mu^- = d\mu_t^- dt$

$$S_{t_0} \equiv S \bigcap \{(x,t);\; t = t_0\}$$

consists of at most finitely many points

and

(1) $$\mu_t = \sum_{x_j(t)\in S_t} \alpha_j(t)\delta_{x_{j(t)}} + u(\cdot,t)\, dx \qquad \text{a.e. } t \in [0,\infty)$$

(2) $$\mu_t^- = \sum_{x_j(t)\in S_t} \beta_j^-(t)\delta_{x_{j(t)}} + u(\cdot,t)\, dx \qquad \text{a.e. } t \in [0,\infty)$$

Here $\alpha_j(t) \geq \beta_j^-(t) \geq 0$ for all $0 < t < \infty$

In particular, $\alpha_j(t)=\beta_j^-(t)=0$ for all $0 < t < T_*$

In addition, $u \in L^\infty(0,\infty; L^1(\Omega))$ with

$$\int_\Omega u(x,t)\, dx \leq \int_\Omega u_0(x)\, dx \qquad \text{for all } 0 < t < \infty$$

Theorem 2 [Luckhaus-S-Velázquez.]

Suppose that $\exists\, A \subset [T_*, \infty)$ of positive measure

s.t. $\beta_j^-(t) > 8\pi$ for a.e. $t \in A$ with some j in (2)

$\Rightarrow$ $\alpha_j(t) > \beta_j^-(t)$ for a.e. $t \in A$ in (1) and (2)

$$u^{\varepsilon_k} \to \mu, \qquad f_{\varepsilon_k}(u^{\varepsilon_k}) \to \mu^-, \qquad \alpha_j(t) \geq \beta_j^-(t) \geq 0$$

for all $0 < t < \infty$

(1) $$\mu_t = \sum_{x_j(t) \in S_t} \alpha_j(t) \delta_{x_{j(t)}} + u(\cdot, t)\, dx \qquad \text{a.e. } t \in [0, \infty)$$

(2) $$\mu_t^- = \sum_{x_j(t) \in S_t} \beta_j^-(t) \delta_{x_{j(t)}} + u(\cdot, t)\, dx \qquad \text{a.e. } t \in [0, \infty)$$

Regularization: II

$$
(\mathrm{KS})_{\varepsilon}^{2}
\begin{cases}
u_t = \Delta(u + \varepsilon u^{7/6}) - \operatorname{div}\left(u \nabla v\right), & x \in \Omega, \quad t > 0 \\
-\Delta v = u - \dfrac{1}{|\Omega|} \displaystyle\int_{\Omega} u\,dx, & x \in \Omega, \quad t > 0 \\
\partial_{\nu} u\left(x,t\right) = 0, \quad \partial_{\nu} v\left(x,t\right) = 0 & x \in \partial\Omega, \quad t > 0 \\
u\left(x,0\right) = u_0\left(x\right), & x \in \Omega.
\end{cases}
$$

Theorem 3 [Luckhaus-S-Válazquez.]

$u^\varepsilon \le u^\varepsilon + \varepsilon (u^\varepsilon)^{7/6}$

$\exists u^\varepsilon$: sol. of $(KS)_\varepsilon^2$ on $[0,\infty)$,

$\exists$Radon measure $\tilde{\mu}$, $\mu^+ \in M^+(\Omega \times [0,\infty))$ and $\exists \{\varepsilon_k\}_{k=1}^\infty$

s.t. $u^{\varepsilon_k} \to \tilde{\mu}$, $u^{\varepsilon_k} + \varepsilon_k (u^{\varepsilon_k})^{7/6} \to \mu^+$ in the weak-* topology

Moreover, and $\tilde{\mu} \le \mu^+$

$$d\tilde{\mu} = d\tilde{\mu}_t dt, \quad d\mu^+ = d\mu_t^+ dt$$

with $\tilde{\mu}_t(\Omega) = \int_\Omega u_0(x)\, dx$, $\tilde{\mu}_t \le \mu_t^+$ for all $0 < t < \infty$

Moreover,

the singular set S of $\tilde{\mu}, \mu^+ \subset \overline{\Omega} \times [T_*, \infty)$ for some $0 < T_* < \infty$

with $\tilde{\mu} = \overline{\mu} + \tilde{u}$, $\mu^+ = \overline{\mu}^+ + \tilde{u}$,

where $\overline{\mu}$, $\overline{\mu}^+ \in M^+(\Omega \times [0,\infty))$

and $\tilde{u} \in C^\infty(\Omega \times [0,\infty) \setminus S)$ satisfying (KS) in $\overline{\Omega} \times [0, T_*)$

Moreover, for a.e. $t_0 \in [0,\infty)$ $d\tilde{\mu} = d\tilde{\mu}_t dt,\ \ d\mu^+ = d\mu_t^+ dt$

$$S_{t_0} \equiv S \bigcap \{(x,t);\ t = t_0\}$$

consists of at most finitely many points

and

(3) $$\tilde{\mu}_t = \sum_{x_j(t)\in S_t} \tilde{\alpha}_j(t)\delta_{x_{j(t)}} + \tilde{u}(\cdot,t)\,dx$$ a.e. $t\in[0,\infty)$

(4) $$\mu_t^+ = \sum_{x_j(t)\in S_t} \beta_j^+(t)\delta_{x_{j(t)}} + \tilde{u}(\cdot,t)\,dx$$ a.e. $t\in[0,\infty)$

Here $0 \le \tilde{\alpha}_j(t) \le \beta_j^+(t)$ for all $0 < t < \infty$

In particular, $\tilde{\alpha}_j(t)=\beta_j^+(t)=0$ for all $0 < t < T_*$

In addition, $\tilde{u} \in L^\infty(0,\infty;L^1(\Omega))$ with

$$\int_\Omega \tilde{u}(x,t)\,dx \le \int_\Omega u_0(x)\,dx$$ for all $0 < t < \infty$

Theorem 4 [Luckhaus-S-Velázquez.]

Suppose that $\exists\, A \subset [T_*, \infty)$ of positive measure

s.t. $\tilde{\alpha}_j(t) > 8\pi$ for a.e. $t \in A$ with some j in (4)

⇒ $\beta_j^+(t) > \tilde{\alpha}_j(t)$ for a.e. $t \in A$ in (3) and (4)

$$\tilde{\mu}_t = \sum_{x_j(t)\in S_t} \tilde{\alpha}_j(t)\delta_{x_{j(t)}} + \tilde{u}(\cdot,t)\,dx, \quad \mu_t^+ = \sum_{x_j(t)\in S_t} \beta_j^+(t)\delta_{x_{j(t)}} + \tilde{u}(\cdot,t)\,dx$$

cf

Theorem 2 [Luckhaus-S-Velázquez.]

Suppose that $\exists\, A \subset [T_*, \infty)$ of positive measure

s.t. $\beta_j^-(t) > 8\pi$ for a.e. $t \in A$ with some j in (2)

⇒ $\alpha_j(t) > \beta_j^-(t)$ for a.e. $t \in A$ in (1) and (2)

Remark 2

- $\forall 0 < t < T_*$ Regularization: I = Regularization: II
- $\forall t \ge T_*$ non-uniquenss of measure valued sol

Regularization: I $(\mu_t)_t = \Delta \mu_t + Q[\mu_t^-]$

Regularization: II $(\tilde{\mu}_t)_t = \Delta \mu_t^+ + Q[\tilde{\mu}_t]$

$$\iint Q[\mu_t]\psi\, dx dt$$
$$\approx \frac{1}{4\pi}\iint_{|x-y|>0} \mu_t(x)\mu_t(y) \frac{[(x-y)\cdot(\nabla\psi(x,t)-\nabla\psi(y,t))]}{|x-y|^2} dt$$
$$+\frac{1}{4\pi}\iint_{\Omega\times S^1} (\nu\cdot\nabla^2\psi(x,t)\cdot\nu)\, d\hat{\mu}_t(x,\nu)\, dt$$

$\psi \in C_0^\infty(\Omega\times(0,\infty))$

Theorem 5 [Luckhaus-S-Velázquez.]

continuity for the singular set

$(x_0,t_0)\in S$, we have $\int_{B_R(x_0)\setminus\{x_0\}} d\mu_{t_0} \leq \frac{m_0}{2}$

for some m_0 and $R>0$ fixed.

⇒

$\exists c>0,\ \exists L>0$ depending only on $\|u_0\|_{L^1(\Omega)}$ and Ω

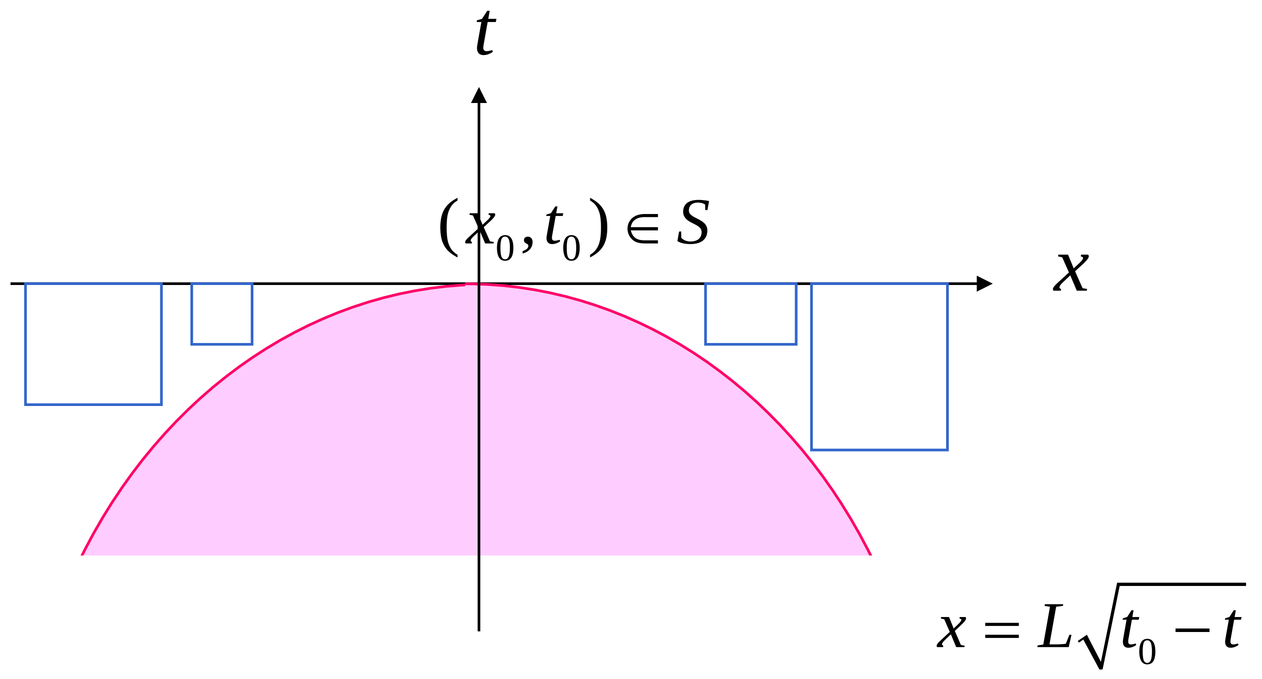
such that

$$S_t \cap B_R(x_0) \subset B_{L\sqrt{t_0-t}}(x_0)$$

for $t\in[t_0-cR^2, t_0]$

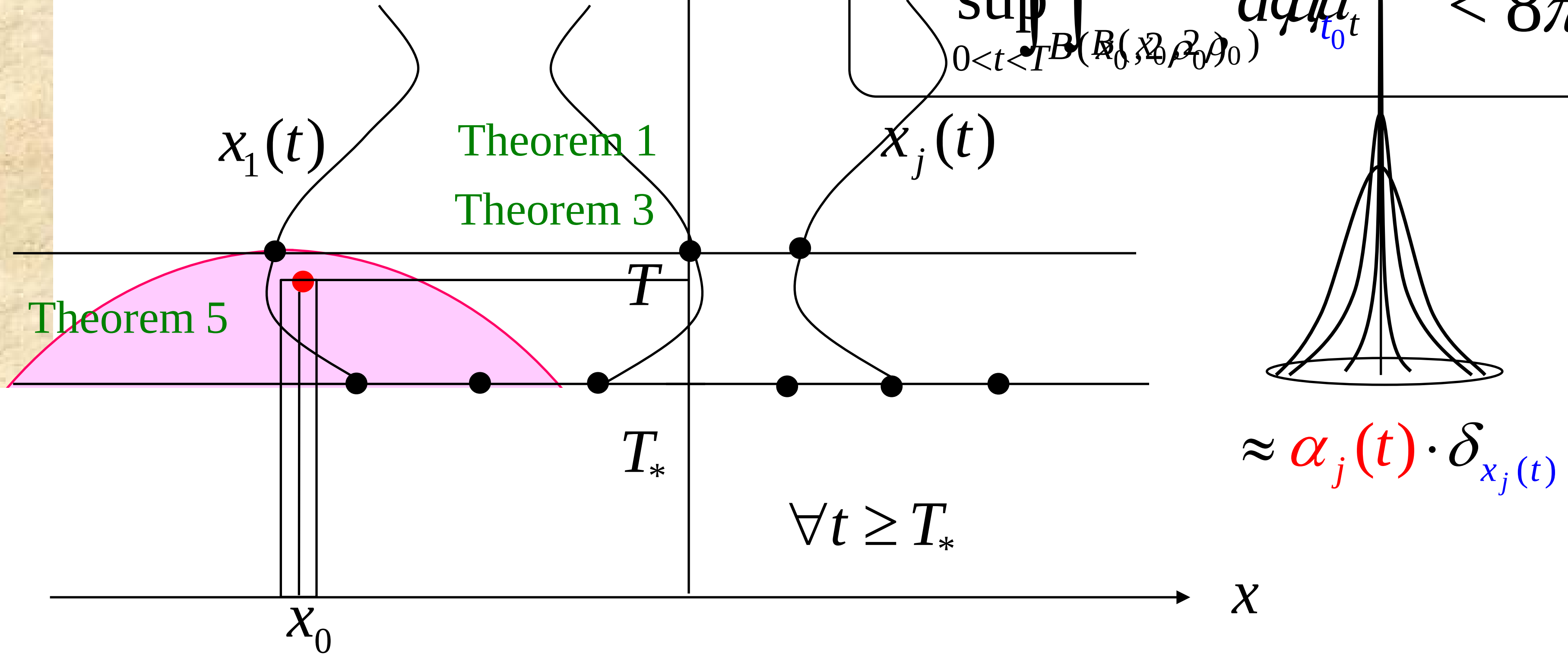
Remark 3

Key idea: ε-regularity thm



$$S_t \cap B_R(x_0) \subset B_{L\sqrt{t_0 - t}}(x_0) \qquad \text{for } t \in [t_0 - cR^2, t_0]$$

IV. Sketch of Proof



$$\sup_{0<t<T} \iint_{B(x_0, 2\rho_0)} d\mu_t < 8\pi$$

$$\approx \alpha_j(t) \cdot \delta_{x_j(t)}$$

$$u^{\varepsilon_k} \to \mu, \quad d\mu = d\mu_t dt$$

(*i*) $\mu_t = \sum_{x_j(t) \in S_t} \alpha_j(t) \delta_{x_j(t)} + u(\cdot, t)\, dx$ a.e. $t \in [0, \infty)$

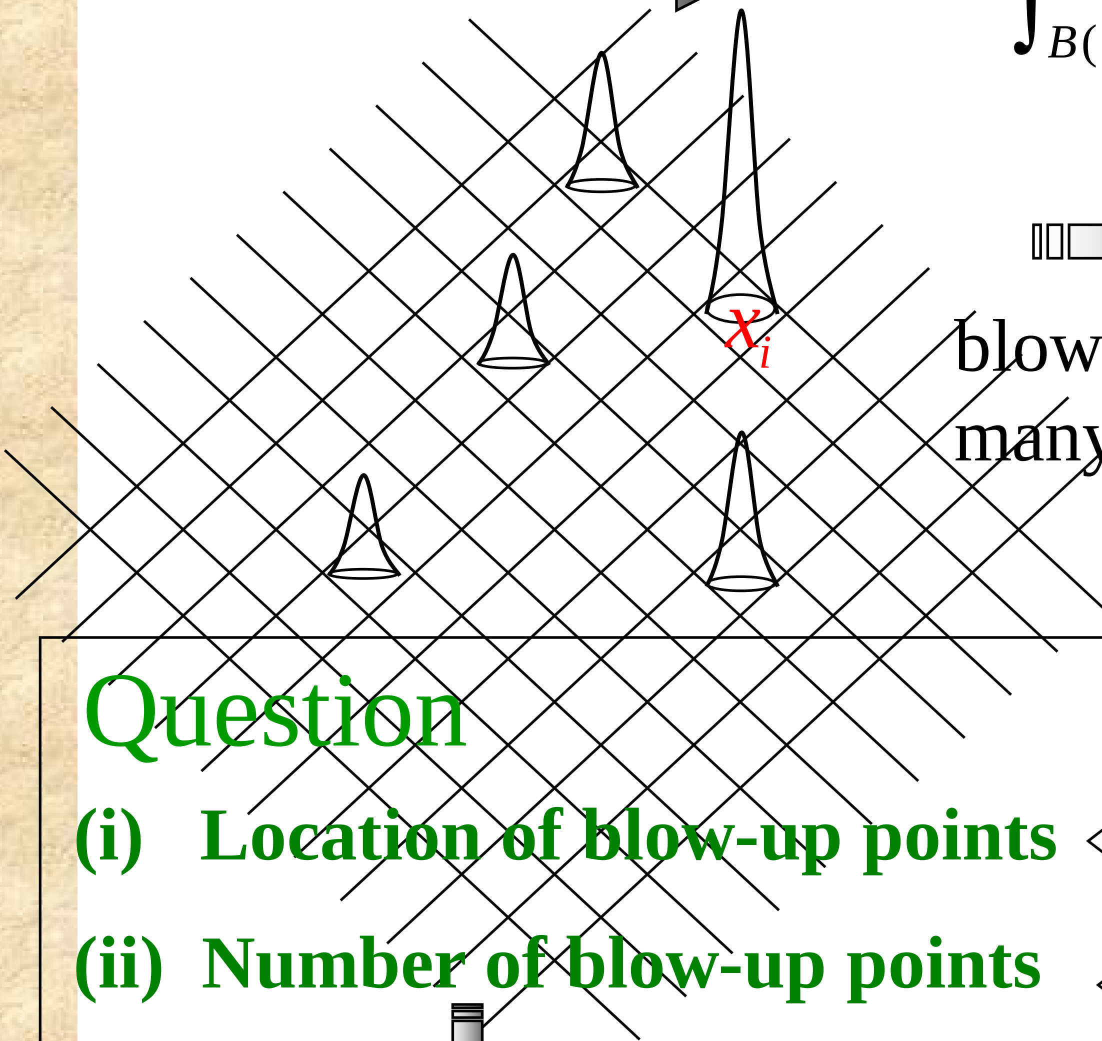
(*ii*) $u \in L^\infty(0, \infty; L^1(\Omega))$

(*iii*) $S_{t_0} \equiv S \cap \{(x,t);\ t = t_0\}$ consists of at most finitely many points

$x_i \in R^N$: blow-up point of u at the time t_0

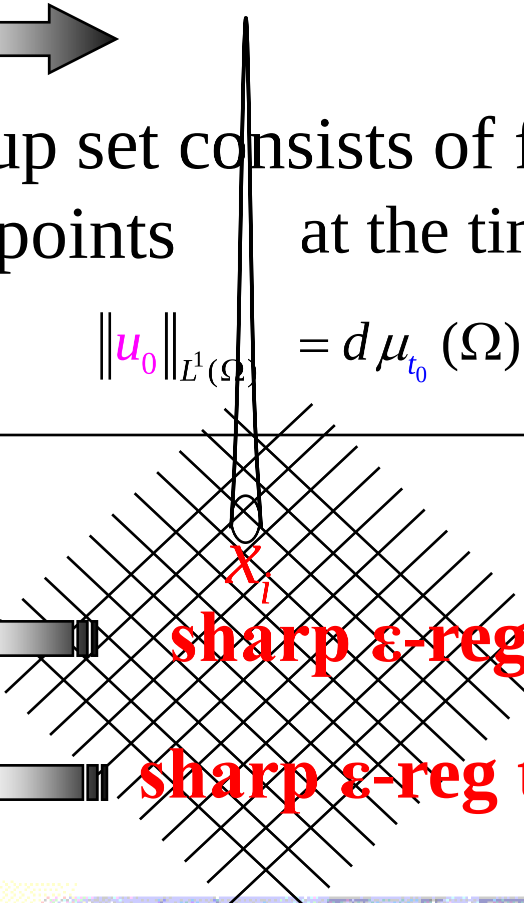
$$\int_{B(x_i,\rho)} d\mu_{t_0} > 8\pi$$

for $\forall \rho > 0$



blow-up set consists of finitely many points at the time t_0

$$\|u_0\|_{L^1(\Omega)} = d\mu_{t_0}(\Omega)$$



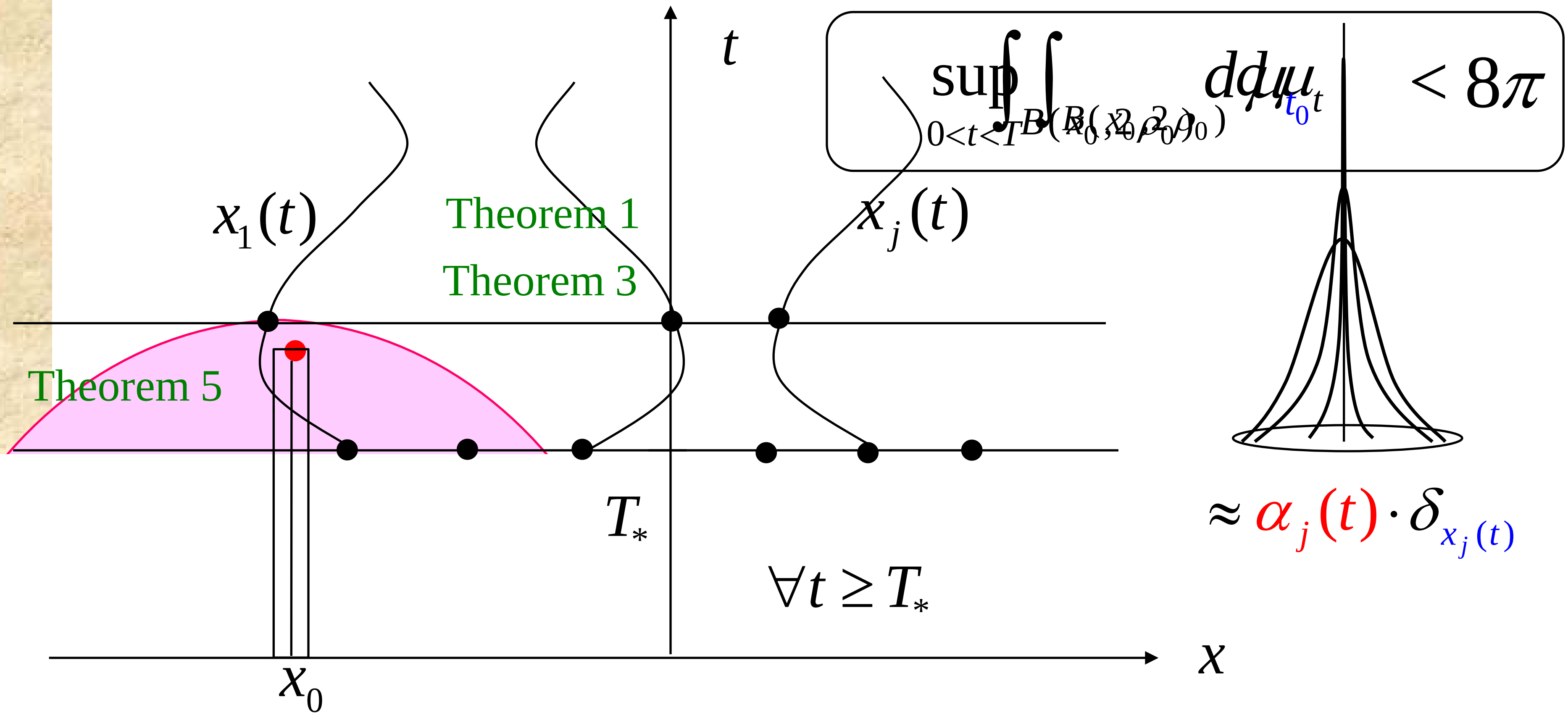
Question

(i) Location of blow-up points ⇐ **sharp ε-reg thm**

(ii) Number of blow-up points ⇐ **sharp ε-reg thm**

⇓

(iii) delta measure



$$u^{\varepsilon_k} \to \mu, \quad d\mu = d\mu_t dt$$

(i) $$\mu_t = \sum_{x_j(t) \in S_t} \alpha_j(t) \delta_{x_j(t)} + u(\cdot, t)\, dx \qquad \text{a.e. } t \in [0, \infty)$$

(ii) $$u \in L^\infty(0, \infty; L^1(\Omega))$$

(iii) $S_{t_0} \equiv S \cap \{(x,t);\ t = t_0\}$ consists of at most finitely many points

Remark 4

Poupaud (Meth. Appl. An. 2002).

Dolbeault-Schmeiser (2006).

complicated assumption on measure

Luckhaus-S-Velazquez

$$\mu_t = \sum_{x_j(t)\in S_t} \alpha_j(t)\delta_{x_{j(t)}} + u(\cdot, t)\, dx, \qquad 0 < t < \infty$$

by ε-regularity thm

Poupaud (Meth. Appl. An. 2002).

Dolbeault-Schmeiser (2006).

$M(R^N)$ is the space of Radon measures,
$M_1(R^N)$ is the space of bounded Radon measures
λ is the Lebesgue measure

Definition

$I \subset R$: interval

$$DM^+(I;R^N) := \{(\mu,\nu);\ \forall t \in I,\ \mu(t) \in M_1^+(R^N),\quad \nu \in M(I \times R^N)^{N\times N},$$

$\mu(t)$ is a tightly continuous with respect to t,
ν is a non negative, symmetric, matrix valued measure,

$$tr(\nu(t,x)) \le \sum_{a \in S_{at}(\mu(t))} (\mu(t)(\{a\}))(\delta(x-a)\lambda(t)\ \}$$

Remark 4

Poupaud (Meth. Appl. An. 2002).

Dolbeault-Schmeiser (2006).

complicated assumption on measure

Luckhaus-S-Velazquez

$$\mu_t = \sum_{x_j(t)\in S_t} \alpha_j(t)\delta_{x_{j(t)}} + u(\cdot,t)\, dx, \qquad 0 < t < \infty$$

by ε-regularity thm

Previous results: (85-95).

Steady states (Schaap).

Finite time blow-up. (Jäger-Luckhaus).

Dirac mass formation. Asymptotics (Herrero-Velazquez).

Critical masses. (Biler, Nagai,...).

Entropies. (Gajewsky-Zacharias).

Multispecies models (Wolansky).

New results: ....

Controlling the motion of the mass:

A crucial symmetrization argument.

Senba-Suzuki (Adv. Diff. Equ. 2001).

Poupaud (Meth. Appl. An. 2002).

Dolbeault-Schmeiser (2006).

Remark 2

- $\forall 0 < t < T_*$ Regularization: I = Regularization: II
- $\forall t \ge T_*$ non-uniquenss of measure valued sol

Regularization: I $\quad (\mu_t)_t = \Delta\mu_t + Q[\mu_t^-]$

Regularization: II $\quad (\tilde{\mu}_t)_t = \Delta\mu_t^+ + Q[\tilde{\mu}_t]$

$$\iint Q[\mu_t]\psi\,dxdt$$

$$\approx \frac{1}{4\pi}\iint_{|x-y|>0} \mu_t(x)\mu_t(y) \frac{[(x-y)\cdot(\nabla\psi(x,t)-\nabla\psi(y,t))]}{|x-y|^2}dt$$

$$+\frac{1}{4\pi}\iint_{\Omega\times S^1}(\nu\cdot\nabla^2\psi(x,t)\cdot\nu)\,d\hat{\mu}_t(x,\nu)\,dt$$

$$\psi \in C_0^\infty(\Omega\times(0,\infty))$$

$$\int_0^\infty \int u_t \cdot \psi \, dx dt = \int_0^\infty \int \Delta u \cdot \psi \, dx dt + \int_0^\infty \int u \nabla v \cdot \nabla \psi \, dx dt$$

($\int_0^\infty \int u \nabla v \cdot \nabla \psi \, dx dt$ is circled in red and marked $= J$.)

$$J = \int_\Omega u(x,t) \int_\Omega \nabla N(x,y)(u(y,t) - \overline{u_\Omega}) dy \nabla \psi(x,t) dx$$

$$\approx \int_\Omega \int_\Omega u(x,t) u(y,t) \frac{[(x-y)\square(\nabla \psi(x,t) - \nabla \psi(y,t))]}{|x-y|^2} dx dy$$

$$= \int_\Omega \int_\Omega \eta\left(\frac{|x-y|}{\delta}\right) \cdots dx dy + \int_\Omega \int_\Omega \left[1 - \eta\left(\frac{|x-y|}{\delta}\right)\right] \cdots dx dy$$

$$\frac{1}{4\pi} \iint_{|x-y|>0} \mu_t(x) \mu_t(y) \; \square \frac{[(x-y)\square(\nabla \psi(x,t) - \nabla \psi(y,t))]}{|x-y|^2} dt$$

$$+ \frac{1}{4\pi} \iint_{\Omega \times S^1} (v \square \nabla^2 \psi(x,t) \square v) d\hat{\mu}_t(x,v) dt \qquad \psi \in C_0^\infty(\Omega \times (0,\infty))$$

Theorem 2 [Luckhaus-S-Velázquez.]

Suppose that $\exists\, A \subset [T_*, \infty)$ of positive measure

s.t. $\beta_j^-(t) > 8\pi$ for a.e. $t \in A$ with some j in (2)

$\Rightarrow$ $\alpha_j(t) > \beta_j^-(t)$ for a.e. $t \in A$ in (1) and (2)

$$u^{\varepsilon_k} \to \mu, \qquad f_{\varepsilon_k}(u^{\varepsilon_k}) \to \mu^-, \qquad \alpha_j(t) \geq \beta_j^-(t) \geq 0$$

for all $0 < t < \infty$

(1) $$\mu_t = \sum_{x_j(t) \in S_t} \alpha_j(t)\delta_{x_{j(t)}}(t) + u(\cdot, t)\, dx \qquad \text{a.e. } t \in [0, \infty)$$

(2) $$\mu_t^- = \sum_{x_j(t) \in S_t} \beta_j^-(t)\delta_{x_{j(t)}}(t) + u(\cdot, t)\, dx \qquad \text{a.e. } t \in [0, \infty)$$

Sketch of proof of Th 2

$$\int_{S^1} d\hat{\mu}_t(x,v) = \sum_{x_j(t)\in S_t} \gamma_j(t)\delta_{x_j(t)}$$

⇒

Fact 1 Fact 2

$$8\pi\beta_j(t) < (\beta_j^-(t))^2 \leq \gamma_j \leq 8\pi\alpha_j(t)$$

assump of Th2

$$\int_{S^1} d\hat{\mu}_t(x,v) \geq (\mu^-{}_{\text{sing}})^2$$

Thank you for your kind attention