プラズマ物理に現れる
Euler-Poisson方程式の定常解について

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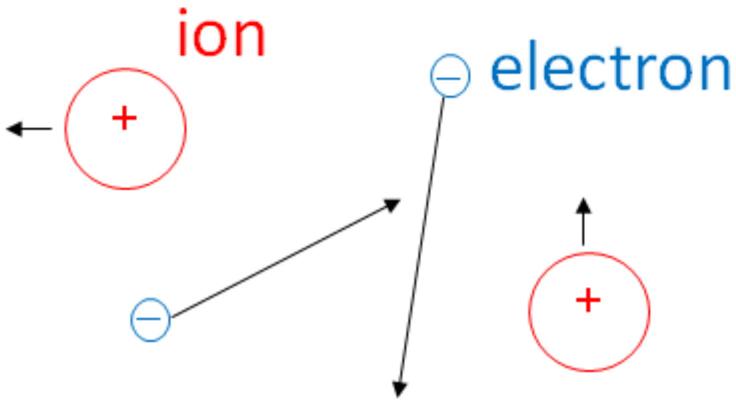
Plasma in Whole Space



$$u_e \gg u_i \ (\because m_e \ll m_i)$$

Nearly neutral : $\rho_e \fallingdotseq \rho_i$
$\phi \fallingdotseq 0$

m : mass

u : velocity

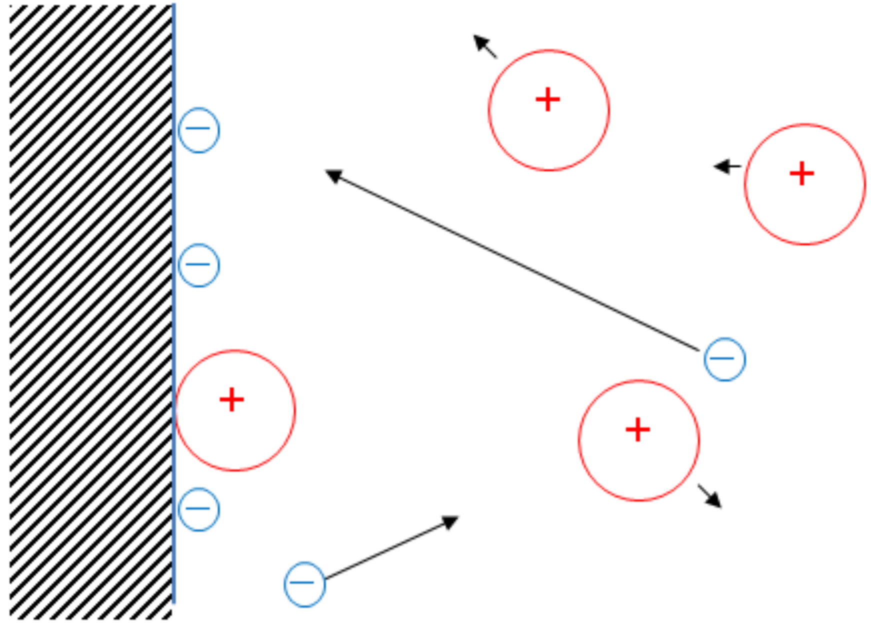
ρ : density

ϕ : electric potential

subscripts
i : ion
e : electron

Put a wall

Plasma in Half Space

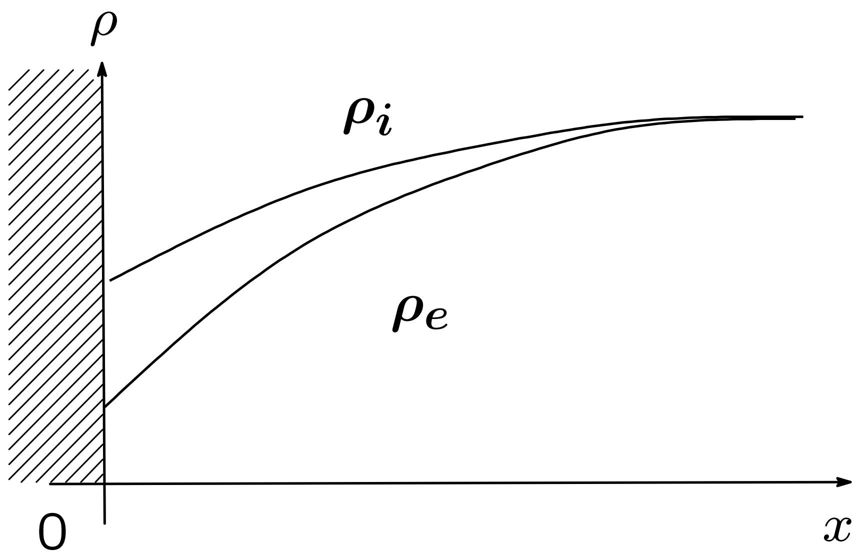


On the wall,
Electrons accumulate

Elsewhere,
Ions dominate

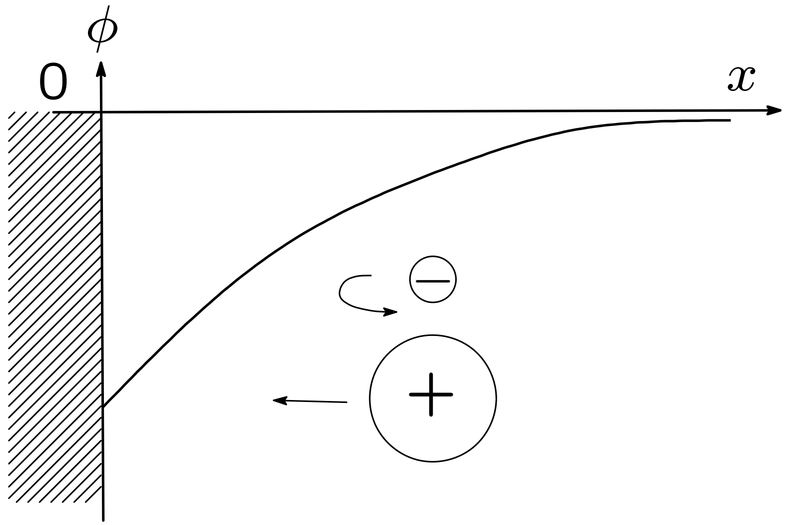
$$(\because u_e \gg u_i)$$

Density



More electrons attain on wall and are absorbed.

Potential



Wall is negatively charged.

<u>Negative potential</u> $\Rightarrow$ Electrons are reflected & ions are accelerated.
$\Rightarrow$ Flux of ions and electrons coincide at wall.
$\Rightarrow$ **It attains steady state.**

This stationary boundary layer is called **sheath**.

<u>**Remark**</u> Density and potential are monotone.

Bohm's Sheath Criterion

For sheath formation, physical observation requires
Bohm sheath criterion :

$$\boldsymbol{u_+^2 > K + 1, \quad u_+ < 0.} \tag{BSC}$$

u_+ : Ion's velocity around sheath edge
K : Ion's temperature ($\sqrt{K}$: sonic velocity)

Remark (BSC) $\Rightarrow$ Supersonic condition : $u_+^2 > K$.

Justify Bohm's physical observation mathematically.

2. Formulation of mathematical problem for $\mathbb{R}_+$

Euler-Poisson equations ($N = 1$)

$$\rho_t + (\rho u)_x = 0, \tag{E.a}$$

$$(\rho u)_t + \left(\rho u^2 + p(\rho)\right)_x = \rho\phi_x, \tag{E.b}$$

$$\phi_{xx} = \rho - \boldsymbol{\rho_e}. \tag{E.c}$$

$x \in \mathbb{R}_+ := (0, \infty),\ t > 0$	:	Space & Time variables
$\rho = \rho(t, x) > 0$	:	Ion density
$u = u(t, x) \in \mathbb{R}$	:	Ion velocity
$\phi = \phi(t, x) \in \mathbb{R}$	:	Electrostatic potential $\times(-1)$
$p(\rho) = K\rho \quad (K > 0)$	:	Pressure (Isothermal)
$\boldsymbol{\rho_e = e^{-\phi} > 0}$	:	**Electron density (Boltzmann relation)**

[Chen, Introduction plasma physics, '77]

introduces the Euler-Poisson equations (E).

• Initial data

$$(\rho, u)(0, x) = (\rho_0, u_0)(x), \tag{I.a}$$

$$\inf_{x \in \mathbb{R}_+} \rho_0(x) > 0, \quad \lim_{x \to \infty} (\rho_0, u_0)(x) = (\rho_+, u_+), \quad \rho_+ > 0, \tag{I.b}$$

where ρ_+, u_+ are constants.

• Boundary data

$$\phi(t, 0) = \phi_b, \tag{B}$$

where ϕ_b is constant.

• Reference point of potential

$$\lim_{x \to \infty} \phi(t, x) = 0. \tag{R}$$

♢ To construct classical solution to (E.c), it is necessary that

$$\rho_+ = 1. \tag{A}$$

Definition

"Sheath" $\Leftrightarrow$ monotone stationary solution with

$$\boldsymbol{u_+^2 > K + 1, \quad u_+ < 0.} \tag{BSC}$$

Stationary problem

Stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$ is solution to (E) independent of t,

$$(\tilde{\rho}\tilde{u})_x = 0, \tag{S.a}$$

$$\left(\tilde{\rho}\tilde{u}^2 + p(\tilde{\rho})\right)_x = \tilde{\rho}\tilde{\phi}_x, \tag{S.b}$$

$$\tilde{\phi}_{xx} = \tilde{\rho} - e^{-\tilde{\phi}} \tag{S.c}$$

with conditions (I.b), (B), (R), (A)

$$\inf_{x \in \mathbb{R}_+} \tilde{\rho}(x) > 0, \qquad \lim_{x \to \infty} (\tilde{\rho}, \tilde{u}, \tilde{\phi})(x) = (1, u_+, 0), \qquad \tilde{\phi}(0) = \phi_b.$$

Problem

1. When does stationary solution exist ?
2. Is sheath asymptotically stable ?

Related results on asymptotic analysis

$\boxed{\text{(E) over bounded domain } (0,1)}$

Existence of stationary solution

- **[A. Ambroso, F. Méhats, P.-A. Raviart, AA'01]**
  Existence of stationary solution is shown under (BSC).

Stability of stationary solution

- **[A. Ambroso M3AS'06]**
  Numerical result. Solution approaches stationary solution.

It is open problem to prove its stability.

Existence of monotone stationary solution

$$(\tilde\rho\tilde u)_x = 0, \quad \left(\tilde\rho\tilde u^2 + K\tilde\rho\right)_x = \tilde\rho\tilde\phi_x, \quad \tilde\phi_{xx} = \tilde\rho - e^{-\tilde\phi} \tag{S}$$

with conditions

$$\inf_{x\in\mathbb{R}_+} \tilde\rho(x) > 0, \quad \lim_{x\to\infty} (\tilde\rho, \tilde u, \tilde\phi)(x) = (1, u_+, 0), \quad \tilde\phi(0) = \phi_b.$$

<u>Derive conditions</u>

- $\int_x^\infty \text{(S.a)}\, dx$, $\lim_{x\to\infty} (\tilde\rho, \tilde u)(x) = (1, u_+) \Rightarrow$

$$\tilde\rho\tilde u = u_+.$$

- $\int_x^\infty \text{(S.b)}/\tilde\rho\, dx$, $\lim_{x\to\infty} (\tilde\rho, \tilde u)(x) = (1, u_+) \Rightarrow$

$$\tilde\phi = f(\tilde\rho), \quad f(\tilde\rho) := K \log\tilde\rho + \frac{u_+^2}{2\tilde\rho^2} - \frac{u_+^2}{2}.$$

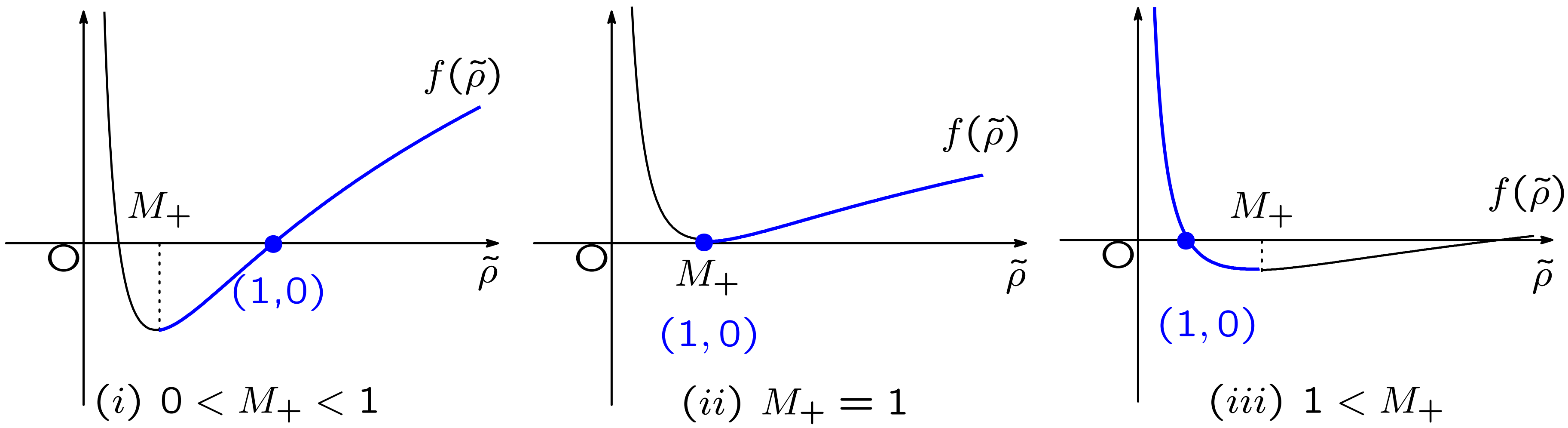
† $\tilde\rho = f^{-1}(\tilde\phi) \Rightarrow$ scalar equation for $\tilde\phi$.

- Define inverse function $f^{-1}(\tilde{\phi})$

  ♢ For $u_+ = 0$, $f^{-1} = e^{\tilde{\phi}/K}$.

  ♢ For $u_+ \neq 0$, define f^{-1} by choosing **blue branch**.

<u>Graph of f</u> **Mach number** $M_+ := |u_+|/\sqrt{K}$. $\lim_{x\to\infty}(\tilde{\rho}, \tilde{\phi}) = (1, 0)$.



- Substitute $\tilde{\rho} = f^{-1}(\tilde{\phi})$ in (S.c), $\int_x^\infty$ (S.c) $\times \tilde{\phi}_x \, dx \Rightarrow$

$$\tilde{\phi}_x{}^2 = 2V(\tilde{\phi}), \quad V(\tilde{\phi}) := \int_0^{\tilde{\phi}} f^{-1}(\eta) - e^{-\eta} \, d\eta,$$

$V(\tilde{\phi})$ is called Sagdeev potential.

Necessary condition for existence

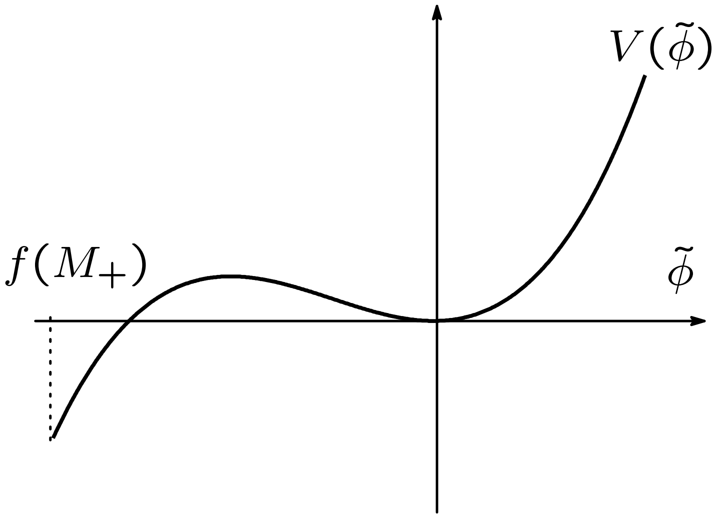
$\diamondsuit$ ϕ_b must belong to image of f, that is,

$$\boldsymbol{\phi_b \geq f(M_+)}.$$

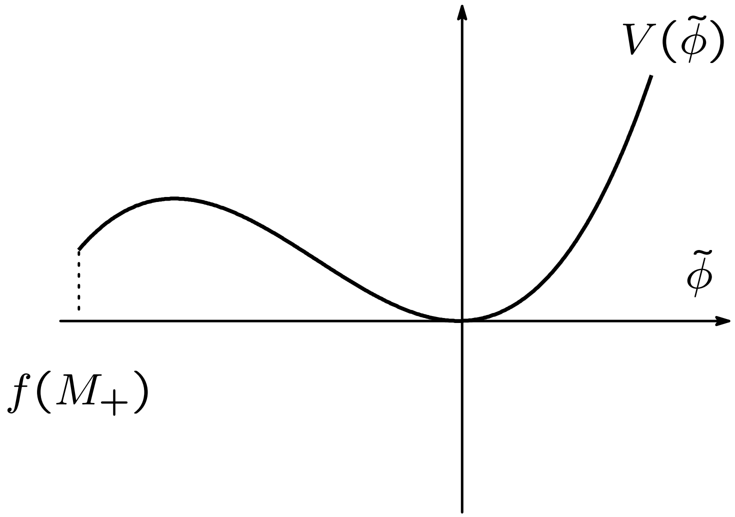
$\diamondsuit$ It is necessary that

$$\boldsymbol{V(\phi_b) \geq 0}.$$

Graph of V Define $\omega \in (K+1, \infty)$ s.t. $\omega \lesseqgtr u_+^2 \iff 0 \lesseqgtr V\left(f(M_+)\right)$.



$\boldsymbol{(i)\ K+1 < u_+^2 \leq \omega}$



$\boldsymbol{(ii)\ \omega < u_+^2}$

Main result $\cdots$ **[M.S. to appear in KRM]**

Theorem 1 (Existence of monotone stationary solution)

$i)$ *Let* $u_+^2 \le K$ *or* $K+1 = u_+^2$ *or* $\boldsymbol{K+1 < u_+^2}$.

$\boldsymbol{\phi_b \ge f(M_+),\ V(\phi_b) \ge 0} \Longleftrightarrow$ *Monotone stationary solution exists.*

Moreover, assume monotonicity $\Rightarrow$ *uniqueness.*

$ii)$ *Let* $K < u_+^2 < K+1$.

No non-trivial stationary solution exists.

(BSC) & $|\phi_b| \ll 1$ give **sufficiency** for existence of sheath.

Remark

- $u_+^2 \in (K+1, \omega] \Rightarrow$ **non-monotone** solution exists (**NOT** unique).

Asymptotic stability of sheath

<u>Perturbation</u>

$$(v, \tilde{v}) := (\log \rho, \log \tilde{\rho}), \quad (\psi, \eta, \sigma)(t, x) := (v, u, \phi)(t, x) - (\tilde{v}, \tilde{u}, \tilde{\phi})(x).$$

Perturbation (ψ, η, σ) satisfies equations

$$\begin{pmatrix} \psi \\ \eta \end{pmatrix}_t + \begin{pmatrix} \eta + \tilde{u} & 1 \\ K & \eta + \tilde{u} \end{pmatrix} \begin{pmatrix} \psi \\ \eta \end{pmatrix}_x = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} \tilde{v} \\ \tilde{u} \end{pmatrix}_x + \begin{pmatrix} 0 \\ \sigma_x \end{pmatrix}, \quad \text{(P.a)}$$

$$\sigma_{xx} = e^{\psi + \tilde{v}} - e^{\tilde{v}} - e^{-(\sigma + \tilde{\phi})} + e^{-\tilde{\phi}}. \quad \text{(P.b)}$$

with initial and boundary data to (P)

$$(\psi, \eta)(0, x) = (\psi_0, \eta_0)(x) := (\log \rho_0 - \log \tilde{\rho}, u_0 - \tilde{u}_0),$$

$$\lim_{x \to \infty} (\psi_0, \eta_0)(x) = (0, 0), \quad \text{(PI)}$$

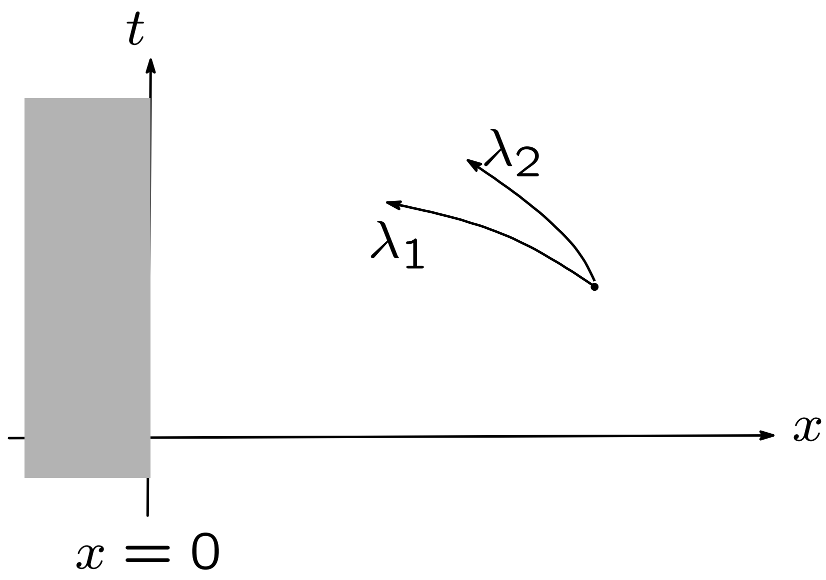
$$\sigma(t, 0) = 0, \quad \lim_{x \to \infty} \sigma(t, x) = 0. \quad \text{(PB)}$$

Around sheath ((BSC) $\Rightarrow$ supersonic)

$\Rightarrow$ Both characteristics of Hyperbolic equations (P.a) are

$$\boldsymbol{\lambda_1 := \eta + \tilde{u} - \sqrt{K} < 0, \quad \lambda_2 := \eta + \tilde{u} + \sqrt{K} < 0.}$$

- No boundary conditions for hyperbolic equations (P.a).
- One boundary condition for elliptic equation (P.b).



$\Rightarrow$ Well-posed with 1 boundary condition (PB),

$$\sigma(t, 0) = 0.$$

Main result $\cdots$ [S.Nishibata, M.Ohnawa, M.S.]

Exponential weight

$$e^{\alpha x/2} \quad \text{for} \quad \alpha > 0.$$

Theorem 2 (Asymptotic stability of sheath)

$$\boldsymbol{u_+ < 0, \quad u_+^2 > K + 1}. \tag{BSC}$$

$\boldsymbol{(e^{\alpha x/2}\psi_0, e^{\alpha x/2}\eta_0) \in H^2(\mathbb{R}_+)}$. $\beta + |\phi_b| + \|(e^{\beta x/2}\psi_0, e^{\beta x/2}\eta_0)\|_2 \ll 1 \Rightarrow$

$^{\exists}1$ *Time global solution* $(e^{\beta x/2}\psi, e^{\beta x/2}\eta, e^{\beta x/2}\sigma) \in \bigcap_{i=0}^{2} C^i([0,\infty); H^{2-i})$.

Moreover, $e^{\beta x/2}\sigma \in C([0,\infty); H^4)$,

$$\|(e^{\beta x/2}\psi, e^{\beta x/2}\eta)(t)\|_2^2 + \|e^{\beta x/2}\sigma(t)\|_4^2 \le C\|(e^{\beta x/2}\psi_0, e^{\beta x/2}\eta_0)\|_2^2\, \boldsymbol{e^{-\gamma t}},$$

C, γ *are positive constants.*

$\|\cdot\|_i := \|\cdot\|_{H^i}$: H^i–Sobolev norm.

Algebraic weight

$$w_{\lambda,\alpha} := (1+\alpha x)^{\lambda/2} \quad \text{for} \quad \lambda > 0,\ \alpha > 0.$$

Theorem 3 (Asymptotic stability of sheath)

$$\boldsymbol{u_+ < 0, \quad u_+^2 > K + 1}. \tag{BSC}$$

$\lambda \geq 2$, $\boldsymbol{(w_{\lambda,\alpha}\psi_0, w_{\lambda,\alpha}\eta_0) \in H^2(\mathbb{R}_+)}$, $\beta + |\phi_b| + \|(w_{\lambda,\alpha}\psi_0, w_{\lambda,\alpha}\eta_0)\|_2 \ll 1$

$\Rightarrow$ $^{\exists 1}$ *Time global solution* $(w_{\lambda,\beta}\psi, w_{\lambda,\beta}\eta, w_{\lambda,\beta}\sigma) \in \bigcap_{i=0}^{2} C^i([0,\infty); H^{2-i})$.

Moreover, $w_{\lambda,\beta}\sigma \in C([0,\infty), H^4)$,

$$\|(w_{\nu,\beta}\psi, w_{\nu,\beta}\eta)(t)\|_2^2 + \|w_{\nu,\beta}\sigma(t)\|_4^2 \leq C\|(w_{\lambda,\beta}\psi_0, w_{\lambda,\beta}\eta_0)\|_2^2 \boldsymbol{(1+\beta t)^{\lambda-\nu}}$$

for $\nu \in (0,\lambda]$, C *is a positive constant.*

$\|\cdot\|_i := \|\cdot\|_{H^i}$: H^i–Sobolev norm.

(BSC) gives **sufficiency** for asymptotic stability of sheath.

4. Results for $\mathbb{R}^2_+$ & $\mathbb{R}^3_+$

Euler-Poisson equations (dimension $N = 2, 3$)

$$\rho_t + \mathrm{div}\,(\rho u) = 0, \tag{E.a}$$

$$(\rho u)_t + \mathrm{div}\,(\rho u \otimes u) + \nabla p(\rho) = \rho \nabla \phi, \tag{E.b}$$

$$\Delta \phi = \rho - \boldsymbol{\rho_e}. \tag{E.c}$$

$t > 0$	:	Time variable
$x = (x_1, x') \in (0, \infty) \times \mathbb{R}^{N-1} =: \mathbb{R}^N_+$	:	Space variables
$\rho = \rho(t, x) > 0$	:	Ion density
$u = u(t, x) \in \mathbb{R}^N$	:	Ion velocity
$\phi = \phi(t, x) \in \mathbb{R}$	:	Electrostatic potential$\times(-1)$
$p(\rho) = K\rho \quad (K > 0)$	:	Pressure (Isothermal)
$\boldsymbol{\rho_e = e^{-\phi} > 0}$ **(Boltzmann relation)**	:	**Electron density**

$$\nabla = \left(\partial_{x_1}, \cdots, \partial_{x_N}\right), \quad (u \otimes u)_{ij} = u_i u_j, \quad \Delta = \left(\partial^2_{x_1} + \cdots + \partial^2_{x_N}\right).$$

• Initial data

$$(\rho, u)(0, x) = (\rho_0, u_0)(x), \tag{I.a}$$

$$\inf_{x \in \mathbb{R}^N_+} \rho_0(x) > 0, \qquad \lim_{x_1 \to \infty} (\rho_0, u_0)(x_1, x') = (\rho_+, u_+, 0, \cdots, 0), \qquad \rho_+ > 0. \tag{I.b}$$

where ρ_+, u_+ are constants.

• Boundary data

$$\phi(t, 0, x') = \phi_b, \tag{B}$$

where ϕ_b is constant.

• Reference point of potential

$$\lim_{x_1 \to \infty} \phi(t, x_1, x') = 0. \tag{R}$$

♢ To construct classical solution to (E.c), it is necessary that

$$\rho_+ = 1. \tag{A}$$

Definition

"Sheath" $\Leftrightarrow$ monotone planar stationary solution with

$$\boldsymbol{u_+^2 > K + 1, \quad u_+ < 0.} \tag{BSC}$$

Stationary problem

Planar stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi}) = (\tilde{\rho}, \tilde{u}_1, 0, \cdots, 0, \tilde{\phi})(x_1)$ is solution to (E) independent of t, x'.

$$(\tilde{\rho}\tilde{u}_1)_{x_1} = 0, \tag{S.a}$$

$$\left(\tilde{\rho}\tilde{u}_1{}^2 + p(\tilde{\rho})\right)_{x_1} = \tilde{\rho}\tilde{\phi}_{x_1}, \tag{S.b}$$

$$\tilde{\phi}_{x_1 x_1} = \tilde{\rho} - e^{-\tilde{\phi}}, \tag{S.c}$$

with conditions (I.b), (B), (R), (A)

$$\inf_{x_1 \in \mathbb{R}_+} \tilde{\rho}(x_1) > 0, \quad \lim_{x_1 \to \infty} (\tilde{\rho}, \tilde{u}_1, \tilde{\phi})(x_1) = (1, u_+, 0), \quad \tilde{\phi}(0) = \phi_b.$$

Theorem 1 (Existence of monotone stationary solution)
(BSC) & $|\phi_b| \ll 1 \Rightarrow$ *Monotone stationary solution* $(\tilde{\rho}, \tilde{u}_1, \tilde{\phi})$ *exists.*

Main result $(N=2,3)$ $\cdots$[S.Nishibata, M.Ohnawa, M.S.]

Exponential weight

$$e^{\alpha x_1/2} \quad \text{for} \quad \alpha > 0.$$

Theorem 4 (Asymptotic stability of sheath)

$$\boldsymbol{u_+ < 0, \quad u_+^2 > K+1}. \qquad \text{(BSC)}$$

$\boldsymbol{(e^{\alpha x_1/2}\psi_0, e^{\alpha x_1/2}\eta_0) \in H^3(\mathbb{R}^N_+)}$. $\beta + |\phi_b| + \|(e^{\beta x_1/2}\psi_0, e^{\beta x_1/2}\eta_0)\|_3 \ll 1$

$\Rightarrow$

$^{\exists 1}$ *Time global solution* $(e^{\beta x_1/2}\psi, e^{\beta x_1/2}\eta, e^{\beta x_1/2}\sigma) \in \bigcap_{i=0}^{3} C^i([0,\infty); H^{3-i})$.

Moreover, $e^{\beta x_1/2}\sigma \in C([0,\infty); H^5)$,

$$\|(e^{\beta x_1/2}\psi, e^{\beta x_1/2}\eta)(t)\|_3^2 + \|e^{\beta x_1/2}\sigma(t)\|_5^2 \le C\|(e^{\beta x_1/2}\psi_0, e^{\beta x_1/2}\eta_0)\|_3^2 \, \boldsymbol{e^{-\gamma t}},$$

C, γ *are positive constants.*

$\|\cdot\|_i := \|\cdot\|_{H^i}$: H^i–Sobolev norm.

Algebraic weight

$$w_{\lambda,\alpha} := (1 + \alpha x_1)^{\lambda/2} \quad \text{for} \quad \lambda > 0, \; \alpha > 0.$$

Theorem 5 (Asymptotic stability of sheath)

$$\boldsymbol{u_+ < 0, \quad u_+^2 > K + 1}. \tag{BSC}$$

$\lambda \geq 2$, $\boldsymbol{(w_{\lambda,\alpha}\psi_0, w_{\lambda,\alpha}\eta_0) \in H^3(\mathbb{R}^N_+)}$, $\beta + |\phi_b| + \|(w_{\lambda,\alpha}\psi_0, w_{\lambda,\alpha}\eta_0)\|_3 \ll 1$

$\Rightarrow$ $^{\exists 1}$ *Time global solution* $(w_{\lambda,\beta}\psi, w_{\lambda,\beta}\eta, w_{\lambda,\beta}\sigma) \in \bigcap_{i=0}^{3} C^i([0,\infty); H^{3-i})$.

Moreover, $w_{\lambda,\beta}\sigma \in C([0,\infty), H^5)$,

$$\|(w_{\nu,\beta}\psi, w_{\nu,\beta}\eta)(t)\|_3^2 + \|w_{\nu,\beta}\sigma(t)\|_5^2 \leq C\|(w_{\lambda,\beta}\psi_0, w_{\lambda,\beta}\eta_0)\|_3^2 \boldsymbol{(1 + \beta t)^{\lambda-\nu}}$$

for $\nu \in (0, \lambda]$, C *is a positive constant.*

$\|\cdot\|_i := \|\cdot\|_{H^i}$: H^i–Sobolev norm.

(BSC) gives **sufficiency** for asymptotic stability of sheath.

Concluding Remarks

- $u_+^2 > K + 1$ (BSC) $\Rightarrow$
  - $\diamondsuit$ Existence of stationary solution, not-unique.
  - $\diamondsuit$ Monotone stationary solution is unique.
  - $\diamondsuit$ Monotone stationary solution is time asymptotically stable.

- Spectrum analysis supports $\begin{cases} \text{(BSC)} \Rightarrow \text{Linearly stable.} \\ \text{Otherwise} \Rightarrow \text{Linearly unstable.} \end{cases}$
  - $\diamondsuit$ (BSC) may be necessary condition for stability.

We call monotone stationary solution as "**sheath**".