Asymptotic stability of boundary layers to the Euler-Poisson equation arising in plasma physics

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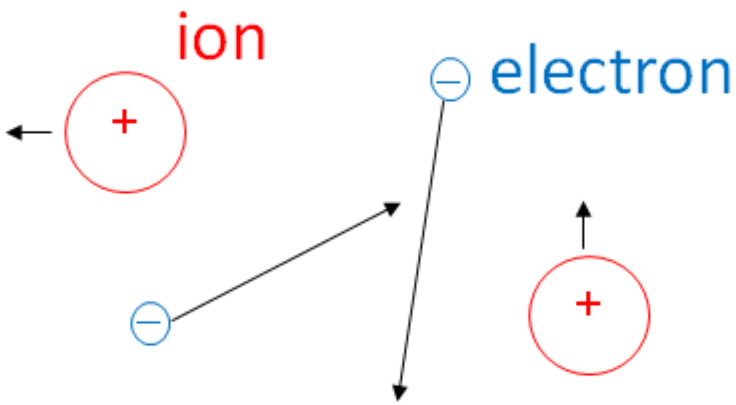
Contents

Process of Sheath Formation(i)

Plasma in Whole Space



$$u_e \gg u_i$$
$$(\because m_e \ll m_i)$$

Nearly neutral : $\rho_e \fallingdotseq \rho_i$
$\phi \fallingdotseq 0$

m : mass

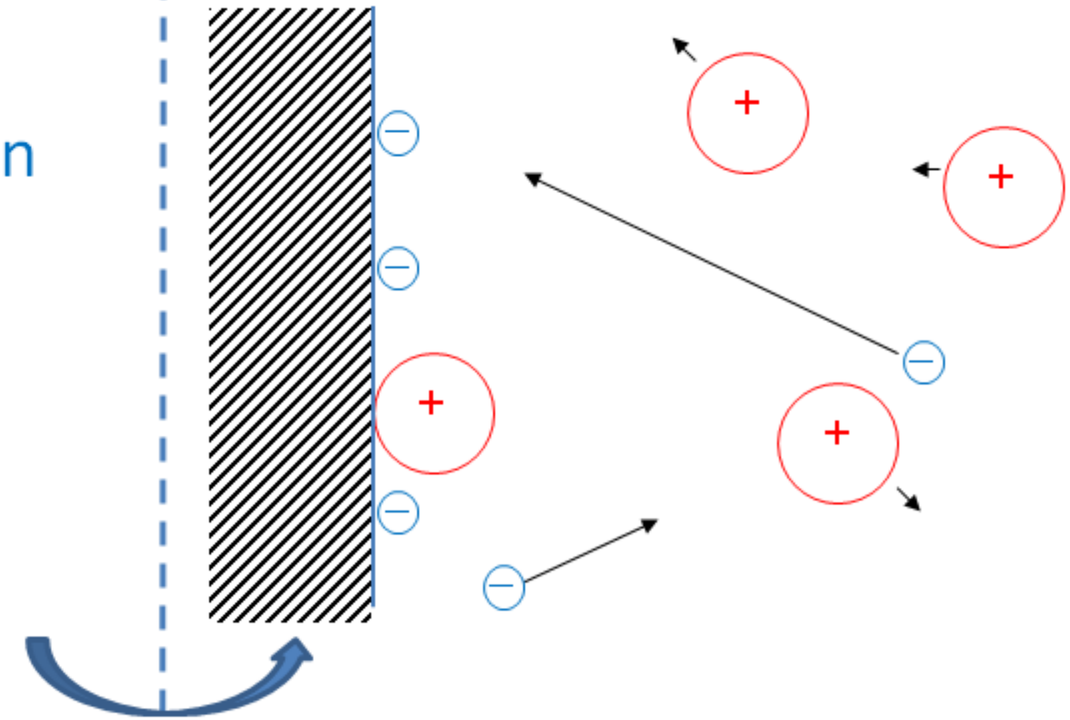
u : velocity

ρ : density

ϕ : electric potential

subscripts
i : ion
e : electron

Plasma in Half Space



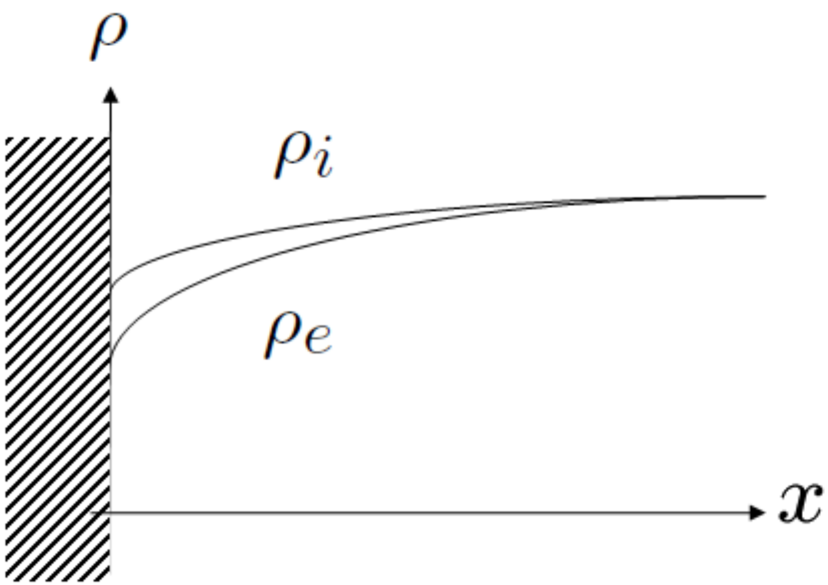
Put a wall

On the wall,
Electrons accumulate
$$(\because u_e \gg u_i)$$

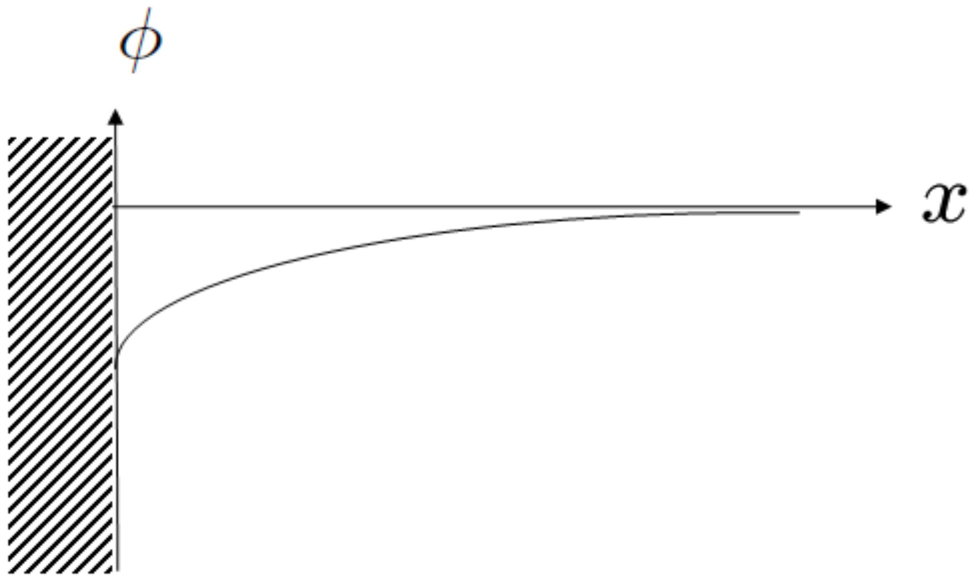
Elsewhere,
Ions dominate

Process of Sheath Formation(ii)

Density distribution



Potential distribution



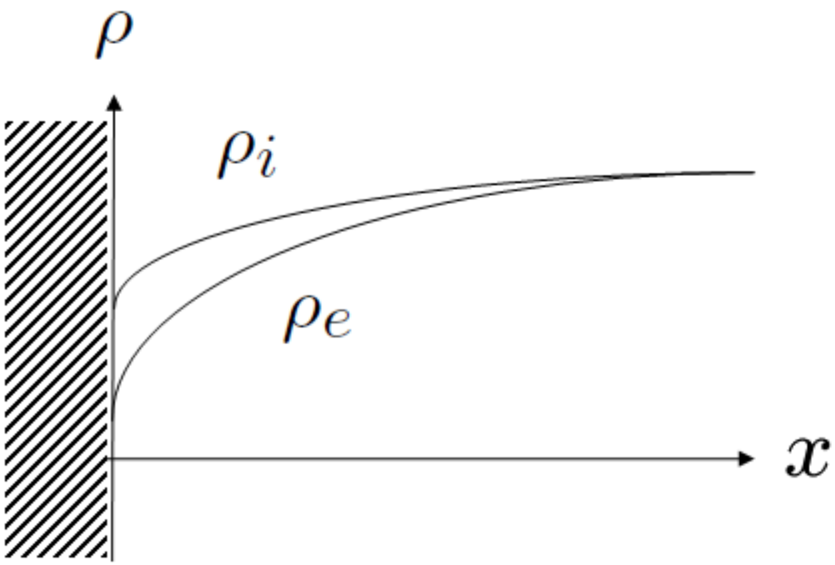
On the wall, electrons gather.
Elsewhere, ions dominate.

Lower potential on the wall.

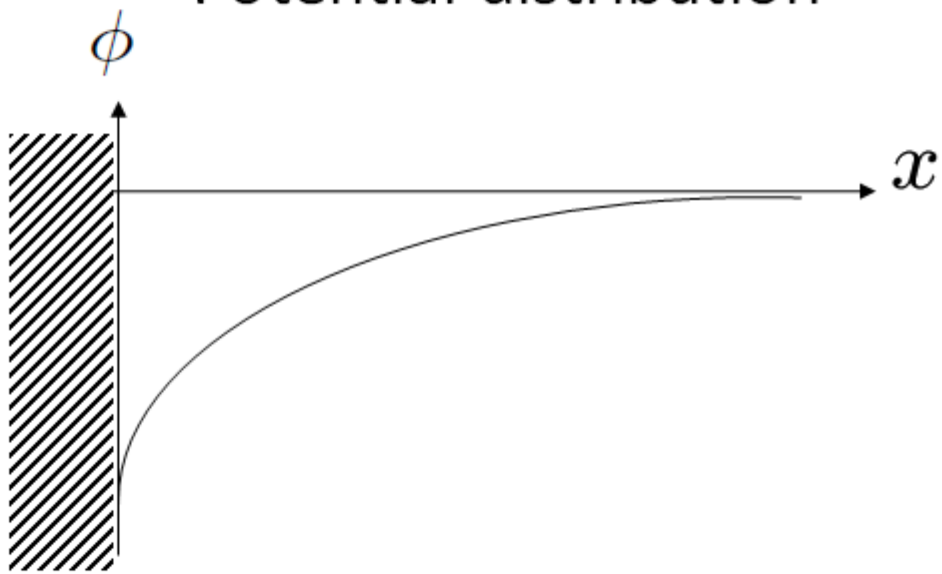
Toward the wall,
ions are accelerated
electrons are decelerated.

Process of Sheath Formation(iii)

Density distribution



Potential distribution



In the end, both flux to the wall coincide
and a steady state is attained.

This <u>stationary boundary layer</u> is called a SHEATH.

Remark : Physically, density & potential are monotone.
We focus our attention on sheath with monotonicity.

Bohm's Sheath Criterion

For the sheath formation, physical observation requires the **Bohm sheath criterion** (BSC):

$$\boldsymbol{u_+^2 \geq K + 1, \quad u_+ < 0,} \tag{BSC}$$

u_+ : Ion's velocity component normal to wall around sheath edge
K : Const. proportional to abs. temperature (= (Acoustic Velocity)^2)

$$(p(\rho) = K\rho, \quad K > 0, \quad \text{Isothermal})$$

$\boxed{\text{Validate BSC from the mathematical point of view.}}$

Remark : (BSC) $\Rightarrow$ <u>Supersonic</u> condition : $u_+^2 > K$.

1. Mathematical formulation of the problem

Euler-Poisson equations (dimension $N = 1, 2, 3$)

$$\rho_t + \mathrm{div}\,(\rho u) = 0, \tag{E.a}$$

$$(\rho u)_t + \mathrm{div}\,(\rho u \otimes u) + \nabla p(\rho) = \rho \nabla \phi, \tag{E.b}$$

$$\Delta \phi = \rho - \boldsymbol{\rho_e}. \tag{E.c}$$

$t > 0$	:	Time variable
$x = (x_1, x') = (x_1, x_2, \ldots, x_N)$ $\in \mathbb{R}^N_+ := (0, \infty) \times \mathbb{R}^{N-1}$	:	Space variables
$\rho = \rho(t, x) > 0$	:	Ion density
$u = u(t, x) \in \mathbb{R}^N$	:	Ion velocity
$\phi = \phi(t, x) \in \mathbb{R}$	:	Electrostatic potential$\times(-1)$
$p(\rho) = K\rho \quad (K > 0)$	:	Pressure (Isothermal)
$\boldsymbol{\rho_e = e^{-\phi} > 0}$ **(Boltzmann relation)**	:	**Electron density**

$$\nabla = \left(\partial_{x_1}, \ldots, \partial_{x_N}\right), \quad (u \otimes u)_{ij} = u_i u_j, \quad \Delta = \left(\partial^2_{x_1} + \cdots + \partial^2_{x_N}\right).$$

[Chen, Introduction to plasma physics, ’77]

• Initial data

$$(\rho, u)(0, x) = (\rho_0, u_0)(x), \tag{I.a}$$

$$\inf_{x \in \mathbb{R}^N_+} \rho_0(x) > 0,$$

$$\lim_{x_1 \to \infty} (\rho_0, u_0)(x_1, x') = (\rho_+, u_+, 0, \ldots, 0) \quad {}^\forall x' \in \mathbb{R}^{N-1}, \tag{I.b}$$

where $\rho_+ (> 0)$, $u_+ (< 0)$ are constants.

• Boundary data

$$\phi(t, 0, x') = \phi_b, \quad {}^\forall x' \in \mathbb{R}^{N-1} \tag{B}$$

where ϕ_b is constant.

• Reference point of potential

$$\lim_{x_1 \to \infty} \phi(t, x_1, x') = 0, \quad {}^\forall x' \in \mathbb{R}^{N-1} \tag{R}$$

♢ To construct classical solution to (E.c): $\Delta\phi = \rho - e^{-\phi}$,
it must be that

$$\rho_+ = 1. \tag{A}$$

Stationary problem

We define sheath by a planar stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})(x_1)$ to (E) independent of $t, x_2, \ldots, x_N$,

$$(\tilde{\rho}\tilde{u})_{x_1} = 0, \tag{S.a}$$

$$\left(\tilde{\rho}\tilde{u}^2 + p(\tilde{\rho})\right)_{x_1} = \tilde{\rho}\tilde{\phi}_{x_1}, \tag{S.b}$$

$$\tilde{\phi}_{x_1 x_1} = \tilde{\rho} - e^{-\tilde{\phi}}, \tag{S.c}$$

with conditions (I.b), (B), (R), (A)

$$\inf_{x_1 \in \mathbb{R}_+} \tilde{\rho}(x_1) > 0, \qquad \lim_{x_1 \to \infty} (\tilde{\rho}, \tilde{u}, \tilde{\phi})(x_1) = (\rho_+, u_+, 0), \qquad \tilde{\phi}(0) = \phi_b.$$

Question

1. When does a stationary solution exist ?
2. Is the stationary solution asymptotically stable ?
3. What is the convergence rate ?

Derivation of the Bohm criterion in physical textbooks

In the stationary problem, by integrating (E.a), (E.b′) and (E.c) over (x, ∞) in spatial coordinate, we have

$$e^v u = u_+, \qquad \text{(GB.a)}$$

$$\frac{u^2}{2} + Kv - \phi = \frac{u_+^2}{2}, \qquad \text{(GB.b)}$$

$$\frac{1}{2}\phi'(x)^2 = \int_\infty^x \phi'(y) e^{v(y)} dy + e^{-\phi(x)} - 1. \qquad \text{(GB.c)}$$

Given $\phi(x)$, solve for $v(x)(< 0)$ using (GB.a), (GB.b).

Substitute $v(\phi)$ in (GB.c).

RHS of (GB.c) must be positive everywhere.

$$\Rightarrow u_+ < -\sqrt{K+1} \ \text{ is necessary.}$$

Derivation of ion acoustic wave speed

Rewrite (E.a) and (E.b) by deviding by ρ. (dimension $N = 1$)

$$v_t + uv_x + u_x = 0, \tag{E.a$'$}$$

$$u_t + uu_x + Kv_x - \phi_x = 0, \tag{E.b$'$}$$

$$\phi_{xx} = e^v - e^{-\phi}. \tag{E.c$'$}$$

Under "quasi-neutrality" assumption, drop (E.c$'$), replace ϕ with $-v$.

$$v_t + uv_x + u_x = 0, \tag{E.a$'$}$$

$$u_t + uu_x + (K+1)v_x = 0. \tag{E.b$''$}$$

Characteristic of this system are $u \pm \sqrt{K+1}$.

$\sqrt{K+1}$ is the phase velocity of the wave supported in the linearized system of (E.a$'$) and (E.b$''$). This wave is called the ion acoustic wave.

2. Previous results on asymptotic analysis

$\boxed{\text{(a) Results over bounded domain } (0,1)}$

- **[A. Ambroso, F. Méhats, P.-A. Raviart, AA '01]**
  shows existence of stationary solution under (BSC).

- **[A. Ambroso M3AS '06]**
  numerically shows stability of stationary solution.

$\boxed{\text{(b) Results over } \mathbb{R}_+}$

- **[M. Suzuki '10]**
  shows existence of monotone stationary solution over $\mathbb{R}_+$
  and its stability.

Results about stationary solution $\cdots$ **[M.Suzuki]**

Theorem 1 (Existence of monotone stationary solution)

i) *Let* $u_+^2 \le K$ *or* $K+1 = u_+^2$ *or* $\boldsymbol{K+1 < u_+^2}$.

$\boldsymbol{\phi_b \ge f(|u_+|/\sqrt{K}),\ V(\phi_b) \ge 0} \iff$ *Monotone stationary sol exists.*

Moreover, assume monotonicity $\Rightarrow$ *unique.*

ii) *Let* $K < u_+^2 < K+1$.

Non-trivial stationary solution does NOT exist.

$$f(\tilde{\rho}) := K \log \tilde{\rho} + \frac{u_+^2}{2\tilde{\rho}^2} - \frac{u_+^2}{2}, \quad V(\tilde{\phi}) := \int_0^{\tilde{\phi}} [f^{-1}(\eta) - e^{-\eta}]\, d\eta. \left(\begin{array}{c} \text{Sagdeev} \\ \text{potential} \end{array} \right)$$

Remark

(A): $u_+^2 > K+1$ & $|\phi_b| \ll 1$ or (B): $u_+^2 = K+1$ & $\phi_b \ge 0$

gives sufficiency for the existence of sheath.

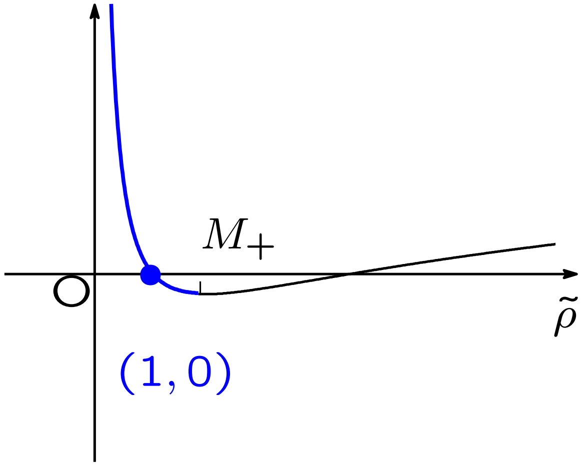
Corollary 2 (Property of sheath)

(A) $\Rightarrow |\partial_{x_1}^j (\tilde{\rho} - \rho_+)(x_1)| + |\partial_{x_1}^j (\tilde{u} - u_+)(x_1)| + |\partial_{x_1}^j \tilde{\phi}(x_1)| \le C|\phi_b| e^{-cx_1}$

Derive conditions for the existence of stationary solution

- $\displaystyle\int_x^\infty (\mathrm{S.b})/\tilde\rho\,dx_1,\ \lim_{x_1\to\infty}(\tilde\rho,\tilde u)(x_1)=(1,u_+)\ \Rightarrow$

$$\tilde\phi=f(\tilde\rho),\qquad f(\tilde\rho):=K\log\tilde\rho+\frac{u_+^2}{2\tilde\rho^2}-\frac{u_+^2}{2}.$$



$f(\rho)$ takes its minimum at $M_+:=|u_+|/\sqrt{K}$. $\Rightarrow\ \phi_b\ge f(M_+)$.
M_+ is the Mach number.

Take the inverse by choosing branch containing asymptotics (1,0) $\Rightarrow$

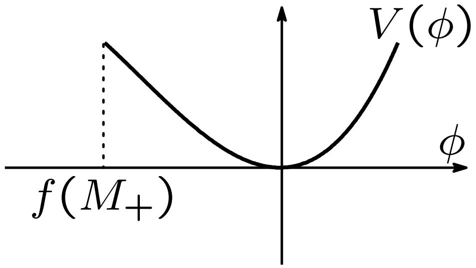
$$\tilde\rho=f^{-1}(\tilde\phi)$$

- Substitute this into (S.c), $\displaystyle\int_x^\infty (\mathrm{S.c})\times\tilde\phi_{x_1}\,dx_1\ \Rightarrow$

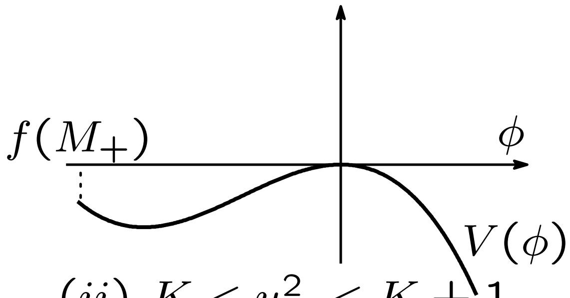
$$\tilde\phi_{x_1}{}^2=2V(\tilde\phi),\qquad V(\tilde\phi):=\int_0^{\tilde\phi}[f^{-1}(\eta)-e^{-\eta}]\,d\eta.\qquad\Rightarrow V(\phi_b)\ge 0.$$

$V(\tilde\phi)$ is the Sagdeev potential.

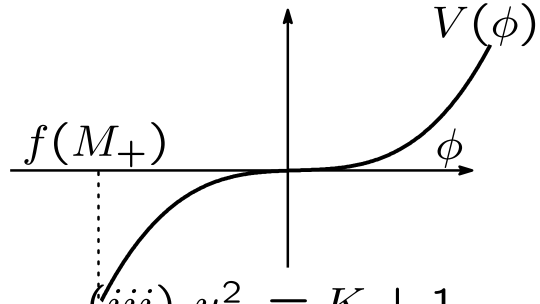
Graph of V



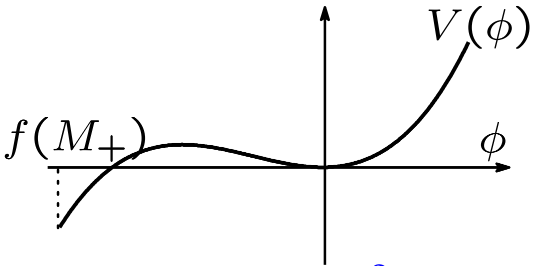
$(i)\ u_+^2 \le K$



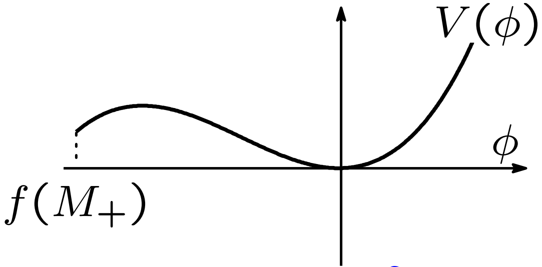
$(ii)\ K < u_+^2 < K+1$



$(iii)\ u_+^2 = K+1$



$\boldsymbol{(iv)\ K+1 < u_+^2 \le \omega}$



$\boldsymbol{(v)\ \omega < u_+^2}$

$$\tilde{\phi}_{x_1}{}^2 = 2V(\tilde{\phi}), \quad \tilde{\phi}(0) = \phi_b, \quad \lim_{x_1 \to \infty} \tilde{\phi}(x_1) = 0,$$

$\sqrt{V}$ is Lipschitz continuous over $[0, \phi_b]$.

$\Rightarrow \phi_b \ge f(|u_+|/\sqrt{K})$ and $V(\phi_b) \ge 0$ gives sufficient condition for the existence of a monotone stationary solution.

Result about asymptotic stability over $\mathbb{R}_+$
(exponential weight) $\cdots$ **[M.Suzuki]**

Perturbation $(\psi, \eta, \sigma)(t, x) := (\log \rho, u, \phi)(t, x) - (\log \tilde{\rho}, \tilde{u}, \tilde{\phi})(x_1)$.

Theorem 3 (Asymptotic stability of sheath)

$$u_+ < 0, \quad u_+^2 > (K + 1/\sqrt{2})^2/K.$$

If $(e^{x_1/2}\psi_0, e^{x_1/2}\eta_0) \in H^2(\mathbb{R}_+)$ *then* $^\exists\epsilon, C, \gamma > 0$*(Const.) s.t.*

$$|\phi_b| + \|(e^{x_1/2}\psi_0, e^{x_1/2}\eta_0)\|_2 \le \epsilon$$

$\Rightarrow$ $^{\exists 1}$ *Time global solution* (ψ, η, σ)

$$e^{x_1/2}\psi, e^{x_1/2}\eta \in \bigcap_{i=0}^{2} C^i\left([0,\infty); H^{2-i}(\mathbb{R}_+)\right),$$
$$e^{x_1/2}\sigma \in \bigcap_{i=0}^{2} C^i\left([0,\infty); H^{4-i}(\mathbb{R}_+)\right),$$

and

$$\|(e^{x_1/2}\psi, e^{x_1/2}\eta)(t)\|_2^2 + \|e^{x_1/2}\sigma(t)\|_4^2 \le C\|(e^{x_1/2}\psi_0, e^{x_1/2}\eta_0)\|_2^2\, e^{-\gamma t}.$$

$\|\cdot\|_i := \|\cdot\|_{H^i}$: H^i–Sobolev norm.

Remark: Condition (1) $\Rightarrow u_+ < 0,\ u_+^2 > K + 1$.

3. Asymptotic stability of sheath in multi-dimensions

Study asymptotic stability of sheath
in multi-dimensional space under exactly (BSC)

Stationary solution

Consider 1D sheath is embedded in multi-dimensional space as

$$\tilde{\tilde{v}}(x) = \tilde{v}(x_1), \quad \tilde{\tilde{u}}(x) = (\tilde{u}(x_1), 0, \ldots, 0), \quad \tilde{\tilde{\phi}}(x) = \tilde{\phi}(x_1). \quad (x \in \mathbb{R}^N_+)$$

Hereafter, we simply write $(\tilde{\tilde{v}}, \tilde{\tilde{u}}, \tilde{\tilde{\phi}})$ by $(\tilde{v}, \tilde{u}, \tilde{\phi})$

Perturbation

$$\begin{aligned}(v, \tilde{v}) &:= (\log \rho, \log \tilde{\rho}),\\ (\psi, \eta, \sigma)(t, x) &:= (v, u, \phi)(t, x_1, x') - (\tilde{v}, \tilde{u}, \tilde{\phi})(x_1).\end{aligned}$$

Perturbation (ψ,η,σ) satisfies equations

$$\begin{pmatrix}\psi\\ \eta\end{pmatrix}_t+\sum_{j=1}^{N} M_j\begin{pmatrix}\psi\\ \eta\end{pmatrix}_{x_j}=-\eta_1\begin{pmatrix}\tilde v\\ \tilde u\end{pmatrix}_{x_1}+\begin{pmatrix}0\\ \nabla\sigma\end{pmatrix}, \tag{P.a}$$

$$\Delta\sigma=e^{\psi+\tilde v}-e^{\tilde v}-e^{-(\sigma+\tilde\phi)}+e^{-\tilde\phi}. \tag{P.b}$$

$$M_j(u_j):=u_j\boldsymbol{I}_{N+1}+\begin{array}{c}\begin{pmatrix} 0 & & 1 & \\ & \mathbf{0} & \vdots & \mathbf{0} \\ K & \cdots & 0 & \cdots \\ & \mathbf{0} & \vdots & \mathbf{0}\end{pmatrix}_{<\,j+1} \\ \widehat{\quad j+1 \quad}\end{array}\quad\begin{pmatrix}\text{matrix of}\\ \text{size (N+1)}\end{pmatrix}$$

with initial and boundary data to (P)

$$(\psi,\eta)(0,x)=(\psi_0,\eta_0)(x):=(\log\rho_0-\log\tilde\rho,u_0-\tilde u),$$
$$\lim_{x_1\to\infty}(\psi_0,\eta_0)(x)=(0,0), \tag{PI}$$
$$\sigma(t,0,x')=0,\qquad \lim_{x_1\to\infty}\sigma(t,x_1,x')=0,\qquad {}^\forall x'\in\mathbb{R}^{N-1}. \tag{PB}$$

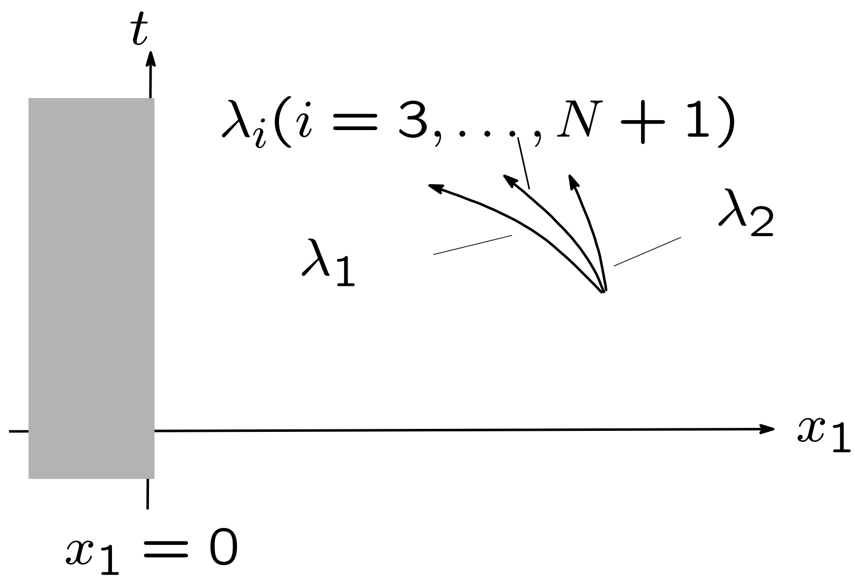
In case perturbation is small ((BSC) $\Rightarrow$ supersonic)

In x_1 direction, characteristics of hyperbolic equations (P.a) are

$$\begin{aligned}\lambda_1 &:= u_1 - \sqrt{K} < 0,\\ \lambda_2 &:= u_1 + \sqrt{K} < 0,\\ \lambda_i &:= u_1 < 0 \quad (i = 3, \ldots, N+1).\end{aligned}$$

$$\left(\because u_1 = u_+ + (\tilde{u} - u_+) + \eta_1, \quad \text{(BSC)} : u_+ \le -\sqrt{K+1}\right)$$

- For hyperbolic equation (P.a), no boundary condition is necessary.
- For elliptic equation (P.b), one boundary condition is necessary.



$\Rightarrow$ Well-posed with 1 boundary condition (PB),

$$\sigma(t, 0, x') = 0, \quad {}^\forall x' \in \mathbb{R}^{N-1}.$$

Difficulty to show asymptotic stability

System of linearized equations of (P)
around asymptotic state $(\rho, u, \phi) = (\rho_+, u_+, 0, \ldots, 0, 0)$ is

$$\begin{pmatrix} \psi \\ \eta \end{pmatrix}_t + \sum_{j=1}^{N} M_j^+ \begin{pmatrix} \psi \\ \eta \end{pmatrix}_{x_j} = \begin{pmatrix} 0 \\ \nabla\sigma \end{pmatrix}, \quad \Delta\sigma = \psi + \sigma. \qquad \text{(L)}$$

$$M_j^+ = M_j(u_+) \quad (j = 1, \ldots, N)$$

Spectrums of (L) are given by

$$\mu(i\xi) = i\left(-\xi_1 u_+ \pm |\xi|\sqrt{K + \frac{1}{1+|\xi|^2}}\right), \quad -i\xi_1 u_+ \,\text{(N-1 multiple)} \quad \xi \in \mathbb{R}^N.$$

Real parts of all spectrums are **ZERO**.

> To resolve this difficulty, we employ **weighted energy method**.

◇ All characteristics go into boundary.

◇ Decay of (ψ_0, η_0) as $x_1 \to \infty \Rightarrow$
convergence of solution towards stationary solution as $t \to \infty$.

Introduce new variables $(\Psi, H, \Sigma) := (e^{\beta x_1/2}\psi, e^{\beta x_1/2}\eta, e^{\beta x_1/2}\sigma)$.

Rewrite systems of equation (P) w.r.t. $(\Psi, H, \Sigma) \Rightarrow (P')$.

Linearize (P') around asymptotic state $(\rho, u, \phi) = (\rho_+, u_+, 0, \dots, 0, 0)$ $\Rightarrow (L')$.

Spectrums of (L') are given by

$$\mu(i\xi) = \frac{\beta u_+}{2} + i\left(-\xi_1 u_+ \pm \sqrt{K\zeta - \frac{1}{\zeta} + 1 - K}\right), \left(\zeta = 1 + |\xi|^2 - \frac{\beta^2}{4} + i\beta\xi_1\right)$$
$$\frac{\beta u_+}{2} - i\xi_1 u_+ \quad \text{(N-1 multiple)} \quad \text{for } \xi \in \mathbb{R}^N.$$

Linearly Stable $\Leftrightarrow \sup_{\xi\in\mathbb{R}^N} Re\,(\mu(i\xi)) < 0 \Leftrightarrow u_+^2 > K + \dfrac{1}{1-\beta^2/4}. \quad (\natural)$

$$\left(\because \quad \sup_{\xi\in\mathbb{R}^N} Re\,(\mu(i\xi)) = Re\,(\mu(0)) \right)$$

$\therefore$ If $u_+^2 > K + 1$, setting $\beta \ll 1$ ensures $(\natural)$.

Main result (with exponential weight)

$\cdots$ **[S.Nishibata, M.O., M.Suzuki]**

Theorem 4 (Asymptotic stability of sheath)

$(N,m)=(1,2)\,,(2,3)\,,(3,3)\,.$

$$u_+<0,\quad u_+^2>K+1,\quad K>0.$$

If $(e^{\lambda x_1/2}\psi_0, e^{\lambda x_1/2}\eta_0)\in H^m(\mathbb{R}^N_+)$ *then* $^\exists\delta>0$ *s.t.*

$$\beta\in(0,\lambda],\quad \beta+\Big(|\phi_b|+\|(e^{\beta x_1/2}\psi_0, e^{\beta x_1/2}\eta_0\|_m\Big)/\beta\le\delta$$

$\Rightarrow\ ^\exists 1$ *Time global solution* (ψ,η,σ)

$$e^{\beta x_1/2}\psi, e^{\beta x_1/2}\eta\ \in\ \bigcap_{i=0}^{m} C^i\Big([0,\infty);H^{m-i}(\mathbb{R}^N_+)\Big),$$

$$e^{\beta x_1/2}\sigma\ \in\ \bigcap_{i=0}^{m} C^i\Big([0,\infty);H^{m+2-i}(\mathbb{R}^N_+)\Big).$$

$$^\exists C>0,\gamma\in(0,\beta]\quad s.t.\quad \|(e^{\beta x_1/2}\psi, e^{\beta x_1/2}\eta)(t)\|_m+\|e^{\beta x_1/2}\sigma(t)\|_{m+2}$$
$$\le C\|(e^{\beta x_1/2}\psi_0, e^{\beta x_1/2}\eta_0)\|_m\, e^{-\gamma t}.$$

Outline of proof (for $N = 2, 3$)

(Local existence) + (A-priori estimate) $\Rightarrow$ (Global existence)

Lemma 5 (Local existence)
$(e^{\lambda x_1/2}\psi_0, e^{\lambda x_1/2}\eta_0) \in H^3(\mathbb{R}^N_+)$ *with*

$$\sup_{x\in\mathbb{R}^N_+}\{\eta_0 + \tilde{u} + \sqrt{K}\} < 0, \quad \sup_{x\in\mathbb{R}^N_+}|\psi_0| + |\phi_b| \ll 1.$$

$\Rightarrow {}^{\exists}\alpha, T > 0$, *s.t.*, ${}^{\exists}1$ *solution* (ψ, η, σ) *s.t.*
$(e^{\alpha x_1/2}\psi, e^{\alpha x_1/2}\eta) \in C([0, T]; H^3(\mathbb{R}^N_+))$, $e^{\alpha x_1/2}\sigma \in C([0, T]; H^5(\mathbb{R}^N_+))$.

$$N_{e^{\alpha x_1}}(T) := \sup_{0\le t\le T}\|(e^{\alpha x_1/2}\psi, e^{\alpha x_1/2}\eta)(t)\|_3.$$

Proposition 6 (A-priori estimate)
$\beta \in (0, \alpha]$, $\beta + (N_{e^{\alpha x_1}}(T) + |\phi_b|)/\beta \ll 1 \Rightarrow {}^{\exists}C, \gamma > 0$ *s.t.*

$$e^{\gamma t}\left(\|e^{\beta x_1/2}(\psi, \eta)(t)\|_3^2 + \|e^{\beta x_1/2}\sigma(t)\|_5^2\right) +$$
$$\int_0^t e^{\gamma\tau}\left(\|e^{\beta x_1/2}(\psi, \eta)(\tau)\|_3^2 + \|e^{\beta x_1/2}\sigma(\tau)\|_5^2\right) d\tau \le C\|e^{\beta x_1/2}(\psi_0, \eta_0)\|_3^2$$

Main result (algebraic weight)

$\cdots$ **[S.Nishibata, M.O., M.Suzuki]**

$$w_{\lambda,\alpha} := (1+\alpha x_1)^{\lambda} \quad \text{for} \quad \lambda > 0,\ \alpha > 0$$

Theorem 7 (Asymptotic stability of sheath)
$(N, m) = (1,2), (2,3), (3,3)$.

$$u_+ < 0, \quad u_+^2 > K+1, \quad K > 0.$$

If $(w_{\lambda/2,\beta}\psi_0, w_{\lambda/2,\beta}\eta_0) \in H^m(\mathbb{R}^N_+)$ *for* $\lambda \geq 2,\ \beta > 0$*, then* $^\forall\alpha \in (0,\lambda]$
$^\exists\delta = \delta(\alpha) > 0$ *s.t. if* $\beta + (|\phi_b| + \|(w_{\lambda/2,\beta}\psi_0, w_{\lambda/2,\beta}\eta_0)\|_m)/\beta \leq \delta$

$\Rightarrow$ $^\exists 1$ *Time global solution* (ψ, η, σ) *s.t.*

$$w_{\alpha/2,\beta}\psi, w_{\alpha/2,\beta}\eta \in \bigcap_{i=0}^{m} C^i([0,\infty); H^{m-i}(\mathbb{R}^N_+)),$$

$$w_{\alpha/2,\beta}\sigma \in \bigcap_{i=0}^{m} C^i([0,\infty); H^{m+2-i}(\mathbb{R}^N_+))$$

$$\textit{and } {}^\exists C(\alpha) > 0 \textit{ s.t.} \quad \|(w_{\alpha/2,\beta}\psi, w_{\alpha/2,\beta}\eta)(t)\|_m^2 + \|w_{\alpha/2,\beta}\sigma(t)\|_{m+2}^2 \leq C\|(w_{\lambda/2,\beta}\psi_0, w_{\lambda/2,\beta}\eta_0)\|_m^2(1+\beta t)^{-(\lambda-\alpha)}.$$

Degenerate case : $u_+^2 = K+1,\ u_+ < 0$

Proposition 8 (Decay rate of stationary sol. in the degenerate case)
Under the degenerate condition,

$${}^\forall\delta_0 > 0,\ {}^\exists C = C(\delta_0) > 0 \textit{ s.t. } {}^\forall x_1 \geq 0 \textit{ and } {}^\forall\phi_b \in (0, \delta_0]$$

$$\left| \partial_{x_1}^i \tilde{\phi}(x_1) \times G(x)^{i+2} - c_i \right| \leq C\phi_b,$$
$$\left| \partial_{x_1}^i \left(\tilde{\rho}(x_1) - \rho_+\right) \times G(x)^{i+2} + c_i \right| \leq C\phi_b,$$
$$\left| \partial_{x_1}^i \left(\tilde{u}(x_1) - u_+\right) / u_+ \times G(x)^{i+2} + c_i \right| \leq C\phi_b,$$

for $i = 0, 1, 2, 3\ldots,$
where $\quad G(x) := \Gamma x_1 + \phi_b^{-1/2}, \quad \Gamma := \sqrt{(K+1)/6}$
and $\quad c_0 := 1,\ c_1 := -2\Gamma,\ c_2 := K+1,\ c_3 := -4\Gamma(K+1), \ldots$

Main result (degenerate case) $\cdots$ [S.Nishibata, M.O., M.Suzuki]

Theorem 9 (Asymptotic stability of sheath)
$(N, m) = (1,2), (2,3), (3,3)$.

$$u_+ < 0, \quad u_+^2 = K + 1, \quad K > 0.$$

Let $\lambda_0 \in \mathbb{R}$ *satisfy* $\lambda_0(\lambda_0 - 1)(\lambda_0 - 2) - 12(\lambda_0 + 2) = 0$, $\lambda \in [4, \lambda_0)$.
${}^\forall\alpha \in (0, \lambda]$, ${}^\forall\theta \in (0,1]$, ${}^\exists\delta = \delta(\alpha, \theta) > 0$
s.t. if $\phi_b \in (0, \delta]$, $\beta/\Gamma\phi_b^{1/2} \in [\theta, 1]$, $(w_{\lambda/2,\beta}\psi_0, w_{\lambda/2,\beta}\eta_0) \in H^m(\mathbb{R}_+^N)$
and $\|(w_{\lambda/2,\gamma}\psi_0, w_{\lambda/2,\gamma}\eta_0)\|_m/\beta^3 \leq \delta$,

$\Rightarrow$ ${}^\exists 1$ *Time global solution* (ψ, η, σ)

$$w_{\alpha/2,\beta}\psi, w_{\alpha/2,\beta}\eta \in \bigcap_{i=0}^{m} C^i([0,\infty); H^{m-i}(\mathbb{R}_+^N)),$$

$$w_{\alpha/2,\beta}\sigma \in \bigcap_{i=0}^{m} C^i([0,\infty); H^{m+2-i}(\mathbb{R}_+^N))$$

$$and\ {}^\exists C(\alpha,\theta) > 0 \quad s.t. \quad \|(w_{\alpha/2,\beta}\psi, w_{\alpha/2,\beta}\eta)(t)\|_m^2 + \|w_{\alpha/2,\beta}\sigma(t)\|_{m+2}^2$$
$$\leq C\|(w_{\lambda/2,\beta}\psi_0, w_{\lambda/2,\beta}\eta_0)\|_m^2(1+\beta t)^{-(\lambda-\alpha)/3}.$$

5. Concluding Remarks

- $u_+^2 \geq K + 1$ (BSC) $\Rightarrow$
  - $\diamondsuit$ Existence of stationary solution, NOT unique
  - $\diamondsuit$ Monotone stationary solution is unique.
  - $\diamondsuit$ Monotone stationary solution is time asymptotically stable

- Spectrum analysis supports $\begin{cases} \text{(BSC)} \Rightarrow \text{Linearly stable.} \\ \text{Otherwise} \Rightarrow \text{Linearly unstable.} \end{cases}$
  - $\diamondsuit$ (BSC) may be a necessary condition for stability.

We call "monotone stationary solution with (BSC)" "**sheath**".