1:2 共鳴によるパターン形成

流体数学セミナー

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by

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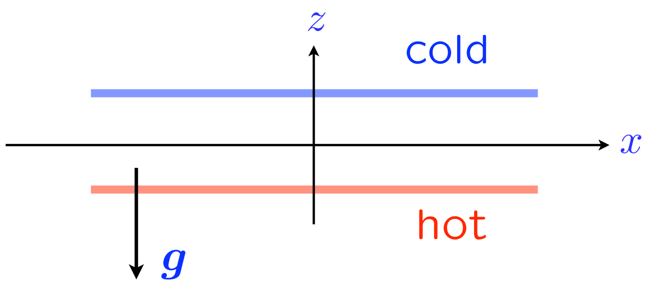
- Background
  - 1:2 resonant interaction under O(2)-symmetry
  - Pattern formation on a hexagonal lattice

    Recent Developments in Nonlinear PDE, 早慶非線形コロキウム
- 1:2 resonance on a hexagonal lattice
  - modified Swift-Hohenberg equation
    - Center manifold reduction
    - Bifurcation diagrams
  - Two-layered Rayleigh-Bénard problem

    京都駅前セミナー，非線形解析セミナー
    - Bifurcation diagrams
    - Nearly heteroclinic cycle

Rayleigh-Bénard problem

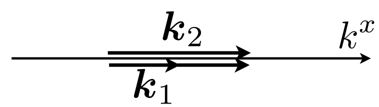


conduction state + disturbance

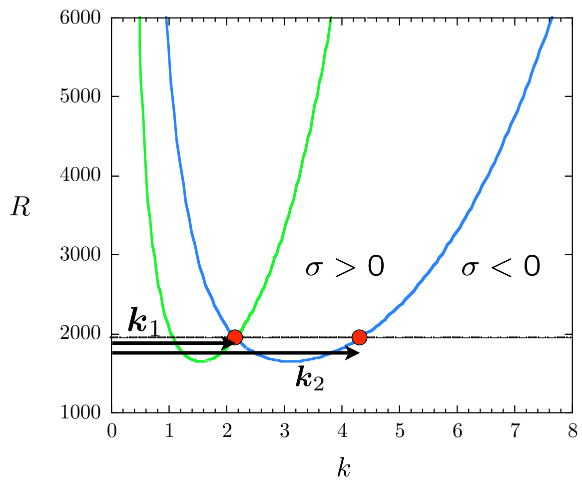
At the linear stage,

$$\text{disturbance} \propto \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}_2+\sigma(\boldsymbol{k},R)t}, \quad \boldsymbol{x}_2=(x,y)$$

$\sigma(\boldsymbol{k},R)=0$: neutral curve



Rayleigh-Bénard problem



1:2 Resonance under O(2) Symmetry

- Classification of Steady Solutions

1. Dangelmayr, G. 1986 Steady state mode interactions in presence of O(2) symmetry. *Dyn. Stab. Syst.* **1**, 159–185.
2. Buzano, E. & Russo, A. 1987 Bifurcation problems with O(2)⊕Z_2 symmetry and the buckling of a cylindrical shell. *Annuli di Matematica Pura ed Applicata (IV)* **146**, 217–262.

1:2 steady-state mode interaction

O(2)=SO(2)×Z_2: $x \to x + l (\text{mod } 2\pi/k)$, $x \to -x$

$$\psi(\boldsymbol{x}, t) = z_1(t)\phi_1(z)\,\mathrm{e}^{ikx} + c.c. + z_2(t)\phi_2(z)\,\mathrm{e}^{2ikx} + c.c + \cdots$$

$$\begin{cases} \dot{z}_1 = \sigma_1 z_1 + \beta_1 \bar{z}_1 z_2 + (\kappa_1 u + \kappa_2 v) z_1, \\ \dot{z}_2 = \sigma_2 z_2 + \beta_2 z_1^2 + (\kappa_3 u + \kappa_4 v) z_2, \end{cases}$$

$$\sigma_1, \sigma_2, \beta_1, \beta_2, \kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \mathbb{R}, \quad u = |z_1|^2, \quad v = |z_2|^2.$$

$$z_1(t) = r_1(t)\,\mathrm{e}^{i\theta_1(t)}, \quad z_2(t) = r_2(t)\,\mathrm{e}^{i\theta_2(t)}.$$

$$\begin{cases} \dot{r}_1 = \sigma_1 r_1 + \beta_1 r_1 r_2 \cos\Theta + (\kappa_1 r_1^2 + \kappa_2 r_2^2) r_1, \\ \dot{r}_2 = \sigma_2 r_2 + \beta_2 r_1^2 \cos\Theta + (\kappa_3 r_1^2 + \kappa_4 r_2^2) r_2, \\ \dot{\Theta} = -(\beta_2 r_1^2 r_2^{-1} + 2\beta_1 r_2)\sin\Theta, \quad \Theta := \theta_2(t) - 2\theta_1(t). \end{cases}$$

Steady solutions

- $r_1 = 0$, $r_2 \neq 0$: pure mode
- $r_1 \neq 0$, $r_2 \neq 0$, $\cos\Theta = \pm 1$: mixed mode

$$\begin{cases} 0 = \sigma_1 r_1 \pm \beta_1 r_1 r_2 + (\kappa_1 r_1^2 + \kappa_2 r_2^2) r_1, \\ 0 = \sigma_2 r_2 \pm \beta_2 r_1^2 + (\kappa_3 r_1^2 + \kappa_4 r_2^2) r_2. \end{cases}$$

- $r_1 \neq 0$, $r_2 \neq 0$, $\cos\Theta \neq \pm 1$: traveling wave

$$\begin{cases} 0 = \sigma_1 r_1 + \beta_1 r_1 r_2 \cos\Theta + (\kappa_1 r_1^2 + \kappa_2 r_2^2) r_1, \\ 0 = \sigma_2 r_2 + \beta_2 r_1^2 \cos\Theta + (\kappa_3 r_1^2 + \kappa_4 r_2^2) r_2, \\ 0 = \beta_2 r_1^2 r_2^{-1} + 2\beta_1 r_2. \end{cases}$$

$\theta_2 - 2\theta_1 = \Theta \neq n\pi$

$\dot{z}_1 = \cdots \Rightarrow \dot{\theta}_1 = \beta_1 r_2 \sin\Theta \;\to\; \theta_1 = (\beta_1 r_2 \sin\Theta)t + \theta_1(0)$, $\theta_2 = 2\theta_1 + \Theta$

$$\psi = r_1\phi_1(z)\,\mathrm{e}^{i(kx+\theta_1)} + c.c. + r_2\phi_2(z)\,\mathrm{e}^{2i(kx+\theta_1)+i\Theta} + c.c. + \cdots$$

$$\mathrm{e}^{i(kx+\theta_1)} \propto \mathrm{e}^{ik(x - [-k^{-1}\beta_1 r_2 \sin\Theta]t)}$$

O(2)⊕ Z_2: $x \to x + l(\text{mod } 2\pi/k)$, $x \to -x$, $z \to -z$

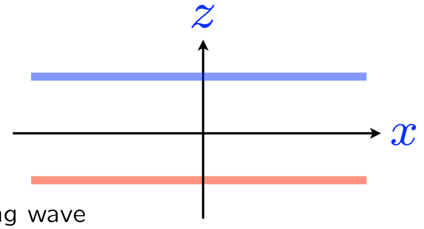
$$\begin{cases} \dot{z}_1 = \sigma_1 z_1 + (\kappa_1 u + \kappa_2 v) z_1 + \chi_1 \bar{z}_1 z_2 w + (\lambda_{11} u^2 + \lambda_{12} uv + \lambda_{13} v^2) z_1, \\ \dot{z}_2 = \sigma_2 z_2 + (\kappa_3 u + \kappa_4 v) z_2 + \chi_2 z_1^2 w + (\lambda_{21} u^2 + \lambda_{22} uv + \lambda_{23} v^2) z_2, \end{cases}$$

$$u = |z_1|^2, \quad v = |z_2|^2, \quad w = \bar{z}_1^2 z_2 + z_1^2 \bar{z}_2.$$

Steady solutions:

$z \to -z : u \to u, w \to -w, \theta \to -\theta$

- $z_1 \neq 0$, $z_2 = 0$: large rolls
- $z_1 = 0$, $z_2 \neq 0$: small rolls
- $z_1 \neq 0$, $z_2 \neq 0$: mixed rolls
- $z_1 \neq 0$, $z_2 \neq 0$, $\Theta \neq n\pi$: traveling wave



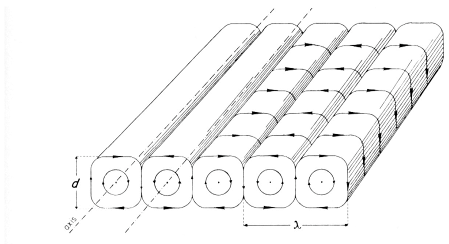
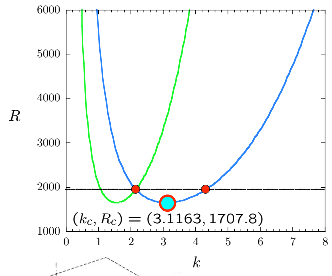


Fig. 2.2. Plane, parallel convection rolls; λ, the dimensional wavelength. After Avsec (1939).



Roll vs. hexgons

Up(L)-hexagons vs.
Down(G)-hexagons

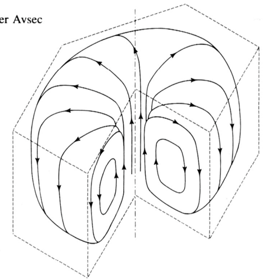
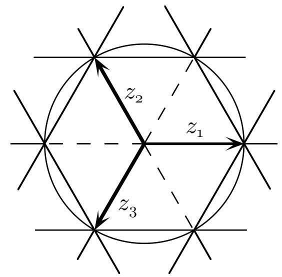


Fig. 2.5. Schematic of the circulation in a hexagonal convection cell in a fluid. After Avsec (1939).



$$\psi(\boldsymbol{x},t) = z_1(t)\phi_1(z)\,\mathrm{e}^{ikx} + c.c. + z_2(t)\phi_2(z)\,\mathrm{e}^{\frac{ik}{2}(x+\sqrt{3}y)} + c.c.$$

$$+ z_3(t)\phi_3(z)\,\mathrm{e}^{\frac{ik}{2}(x-\sqrt{3}y)} + c.c. + \cdots .$$

Pattern formation on a hexagonal lattice

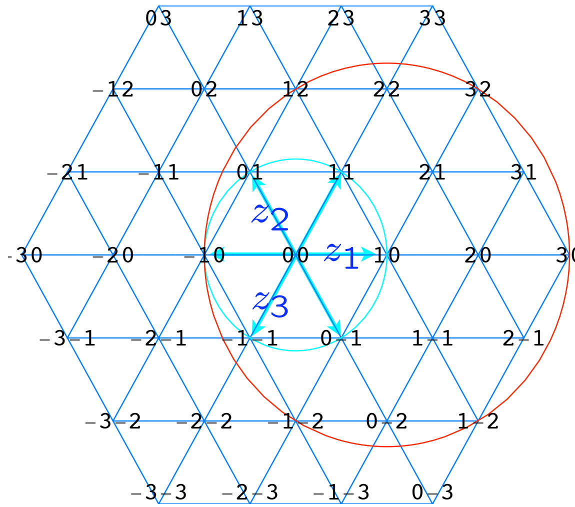
$\Gamma = \mathrm{D}_6 \dot{+} \mathrm{T}^2 \oplus \mathrm{Z}_2$

$z_1, z_2, z_3 \in \mathbb{C}$

$$\mathrm{D}_6 \begin{cases} \mathrm{c}: & (z_1,z_2,z_3) \to (\bar{z}_1,\bar{z}_2,\bar{z}_3) \\ \mathrm{D}_3 \begin{cases} \mathrm{R}_{2\pi/3}: \\ \sigma_v: \end{cases} & \begin{matrix} (z_1,z_2,z_3) \to (z_2,z_3,z_1) \\ (z_1,z_2,z_3) \to (z_1,z_3,z_2) \end{matrix} \end{cases}$$

$$\mathrm{T}^2 : (s,t)\cdot \boldsymbol{z} = (\,\mathrm{e}^{is}z_1,\ \mathrm{e}^{-i(s+t)}z_2,\ \mathrm{e}^{it}z_3),\ \ s,t \in [0,2\pi)$$

$$\mathrm{Z}_2 : (z_1,z_2,z_3) \to (-z_1,-z_2,-z_3)$$



$$\dot z_1 = \sigma z_1 + (\mu_1|z_1|^2 + \mu_2(|z_2|^2+|z_3|^2))z_1 + \nu_1 \bar z_1 \bar z_2^2 \bar z_3^2 + \nu_2 |z_2|^2|z_3|^2 z_1$$

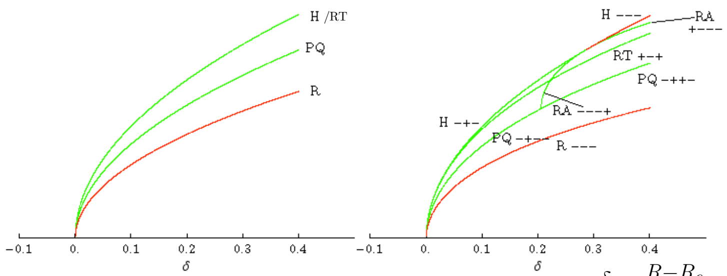
$$+(\kappa_1|z_1|^4 + \kappa_2(|z_2|^2+|z_3|^2)|z_1|^2 + \kappa_3(|z_2|^4+|z_3|^4))z_1$$

Rolls R: $z_1 \neq 0,\ z_2 = z_3 = 0$; Hexagons H: $0 \neq z_1 = z_2 = z_3 \in \mathbb{R}$

Patchwork Quilt PQ: $z_1 = 0,\ z_2 = z_3 \neq 0$; Regular Triangles RT: $0 \neq z_1 = z_2 = z_3 \in i\mathbb{R}$

Rectangles RA: $0 \neq z_1 \neq z_2 = z_3 \neq 0$

KF & S.Yamada: Proc. R. Soc. A (2008) **464** *2721–2739*



$P = 7$ $\qquad \delta = \frac{R - R_c}{R_c}$

Cubic order approx. Quintic order approx.

Swift-Hohenberg equation:

$$\frac{\partial u}{\partial t} = (R - R_c)\sigma_R u - (\Delta + k_c^2)^2 u - f(u), \quad f(u) = u^3$$

$$\frac{\partial u}{\partial t} = ru - (\Delta + 1)^2 u - u^3 \ -\epsilon|\nabla u|^2$$

$u = \delta\, e^{ikx + \sigma t}$:

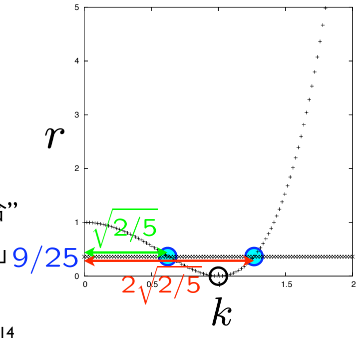
$$\sigma = r - (1 - k^2)^2 + O(\delta^2) \ \Rightarrow \ r = \sigma + (1 - k^2)^2$$

Neutral curve: $\sigma = 0$

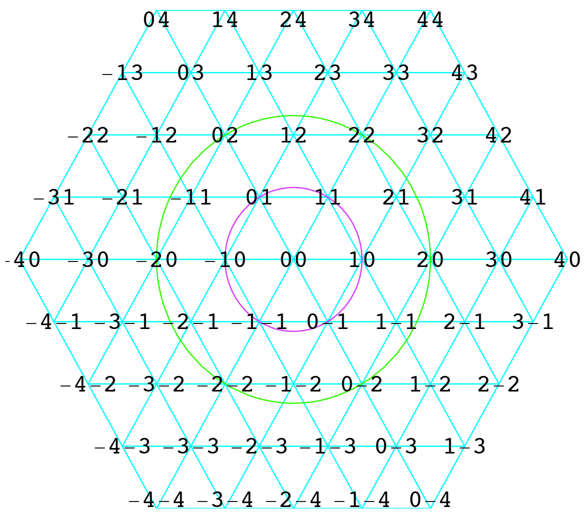
“ロールパターンと六角パターンの競合”

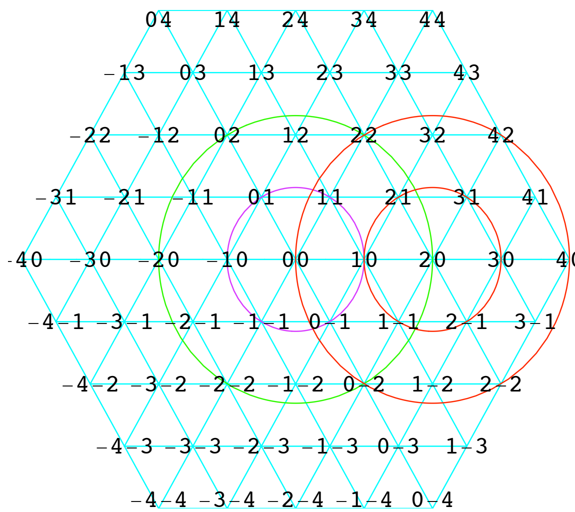
小川知之：「非線形現象と微分方程式」

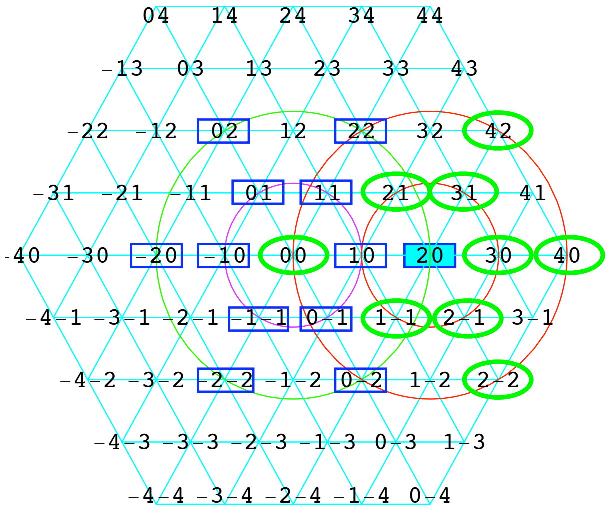
サイエンス社（2010.6）



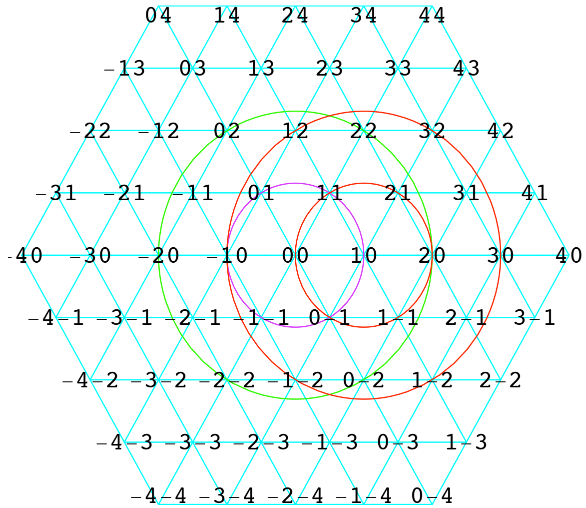
1:2 resonance on a hexagonal lattice



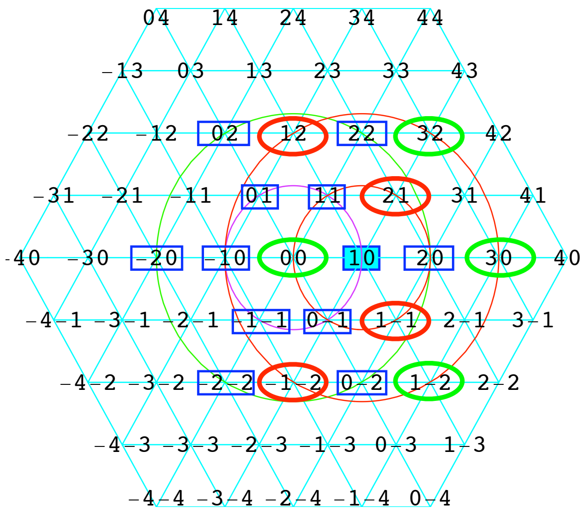



04 14 24 34 44
−13 03 13 23 33 43
−22 −12 02 12 22 32 42
−31 −21 −11 01 11 21 31 41
−40 −30 −20 −10 00 10 20 30 40
−4−1 −3−1 −2−1 −1−1 0−1 1−1 2−1 3−1
−4−2 −3−2 −2−2 −1−2 0−2 1−2 2−2
−4−3 −3−3 −2−3 −1−3 0−3 1−3
−4−4 −3−4 −2−4 −1−4 0−4

17


04 14 24 34 44
−13 03 13 23 33 43
−22 −12 02 12 22 32 42
−31 −21 −11 01 11 21 31 41
−40 −30 −20 −10 00 10 20 30 40
−4−1 −3−1 −2−1 −1−1 0−1 1−1 2−1 3−1
−4−2 −3−2 −2−2 −1−2 0−2 1−2 2−2
−4−3 −3−3 −2−3 −1−3 0−3 1−3
−4−4 −3−4 −2−4 −1−4 0−4

18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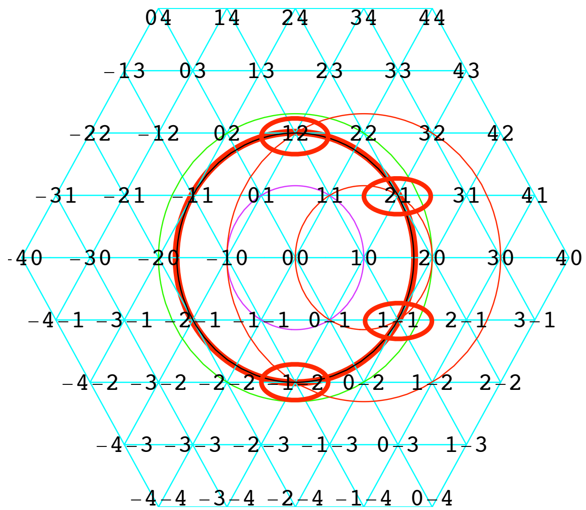
−4−3 −3−3 −2−3 −1−3 0−3 1−3
−4−4 −3−4 −2−4 −1−4 0−4

19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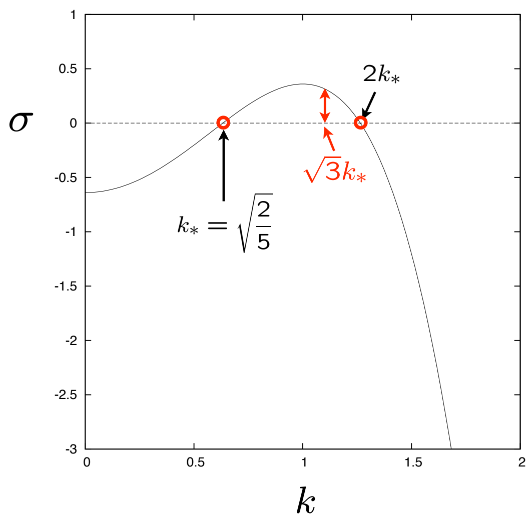
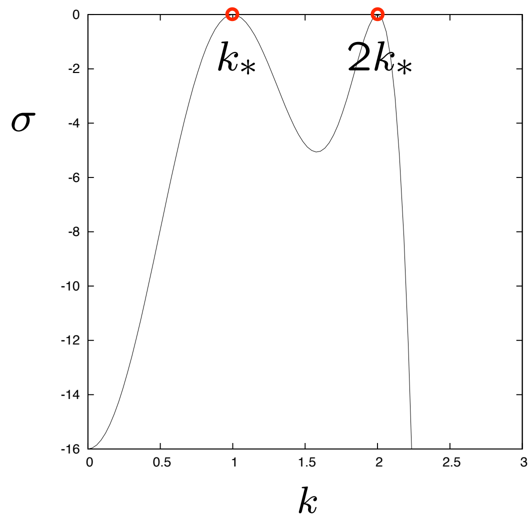
−4−3 −3−3 −2−3 −1−3 0−3 1−3
−4−4 −3−4 −2−4 −1−4 0−4

20

$$\sigma(k) = \frac{9}{25} - (1 - k^2)^2$$

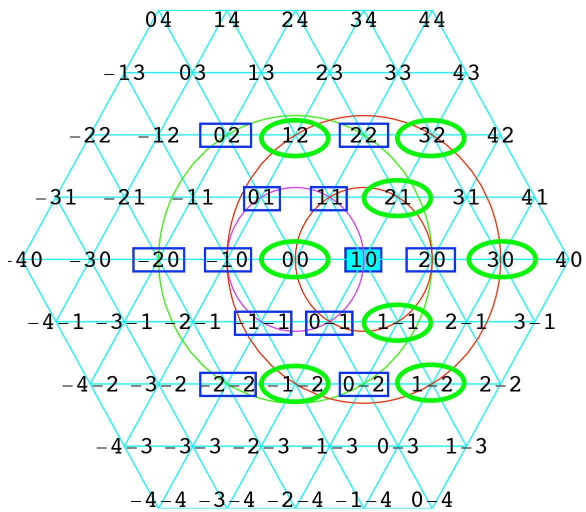


$$\sigma(k) = -(1 - k^2)^2(4 - k^2)^2$$



Modified Swift-Hohenberg equation for 1:2 resonance under broken Z_2-symmetry:

$$\frac{\partial u}{\partial t} = ru - (\Delta + 1)^2(\Delta + 4)^2 u + \epsilon u^2 - u^3$$



Center Manifold Reduction under Broken Z_2-Symmetry

$$u(x,y) = z_{10}\,\mathrm{e}^{ikx} + z_{01}\,\mathrm{e}^{\frac{ik}{2}(x+\sqrt{3}y)} + z_{-1-1}\,\mathrm{e}^{\frac{ik}{2}(x-\sqrt{3}y)} + c.c.$$

$$+ z_{20}\,\mathrm{e}^{2ikx} + z_{02}\,\mathrm{e}^{\frac{2ik}{2}(x+\sqrt{3}y)} + z_{-2-2}\,\mathrm{e}^{\frac{2ik}{2}(x-\sqrt{3}y)} + c.c. + \cdots$$

$$\begin{cases} \dot{z}_{10} = \sigma_{10}z_{10} + 2\epsilon z_{11}z_{0-1} + 2\epsilon z_{20}z_{-10} + O(3), \\ \dot{z}_{01} = \sigma_{01}z_{01} + 2\epsilon z_{11}z_{-10} + 2\epsilon z_{02}z_{0-1} + O(3), \\ \dot{z}_{-1-1} = \sigma_{-1-1}z_{-1-1} + 2\epsilon z_{-10}z_{0-1} + 2\epsilon z_{-2-2}z_{11} + O(3), \\ \dot{z}_{20} = \sigma_{20}z_{20} + \epsilon z_{10}^2 + 2\epsilon z_{22}z_{0-2} + O(3), \\ \dot{z}_{02} = \sigma_{02}z_{02} + 2\epsilon z_{22}z_{-20} + \epsilon z_{01}^2 + O(3), \\ \dot{z}_{-2-2} = \sigma_{-2-2}z_{-2-2} + 2\epsilon z_{-20}z_{0-2} + \epsilon z_{-1-1}^2 + O(3). \end{cases}$$

Center manifold:

$$z_{jk} = h_{jk} = h_{jk}(z_{10}, z_{01}, z_{-1-1}, z_{20}, z_{02}, z_{-2-2}, \bar{z}_{10}, \bar{z}_{01}, \bar{z}_{-1-1}, \bar{z}_{20}, \bar{z}_{02}, \bar{z}_{22})$$

$\sqrt{3}:\ h_{21} = \gamma_{1011}z_{10}z_{11} + \gamma_{220-1}z_{22}z_{0-1} + \gamma_{2001}z_{20}z_{01},$

$\sqrt{3}:\ h_{1-1} = \gamma_{100-1}z_{10}z_{0-1} + \gamma_{20-1-1}z_{20}z_{-1-1} + \gamma_{0-211}z_{0-2}z_{11},$

$\sqrt{7}:\ h_{1-2} = \gamma_{100-2}z_{10}z_{0-2},$

$3:\ h_{30} = \gamma_{1020}z_{10}z_{20},$

$\sqrt{7}:\ h_{32} = \gamma_{2210}z_{22}z_{10},$

$\sqrt{3}:\ h_{12} = \gamma_{1101}z_{11}z_{01} + \gamma_{0210}z_{02}z_{10} + \gamma_{22-10}z_{22}z_{-10},$

$\sqrt{3}:\ h_{-1-2} = \gamma_{-1-10-1}z_{-1-1}z_{0-1} + \gamma_{-2-210}z_{-2-2}z_{10} + \gamma_{-100-2}z_{-10}z_{0-2},$

$h_{00} = \gamma_{10-10}|z_{10}|^2 + \gamma_{010-1}|z_{01}|^2 + \gamma_{-1-111}|z_{-1-1}|^2 + \gamma_{20-20}|z_{20}|^2$

$\quad + \gamma_{020-2}|z_{02}|^2 + \gamma_{-2-222}|z_{-2-2}|^2,$

$4:\ h_{40} = \gamma_{2020}z_{20}^2,$

$2\sqrt{3}:\ h_{42} = \gamma_{2220}z_{22}z_{20},$

$2\sqrt{3}:\ h_{2-2} = \gamma_{200-2}z_{20}z_{0-2},$

$\sqrt{7}:\ h_{31} = \gamma_{2011}z_{20}z_{11},$

$\sqrt{7}:\ h_{2-1} = \gamma_{200-1}z_{20}z_{0-1}.$

$$\gamma_{j_1,j_2,k_1,k_2} = \frac{2\epsilon}{\sigma_{j_1j_2} + \sigma_{k_1k_2} - \sigma_{j_1+k_1,j_2+k_2}}.$$

$$\dot{z}_{10} = \sigma_{10}z_{10} + 2\epsilon z_{11}z_{0-1} + 2\epsilon z_{20}z_{-10} - \left(\frac{4\epsilon^2}{\sigma(0)} + 3\right)u_{10}z_{10}$$

$$-\left(\frac{4\epsilon^2}{\sigma(0)} + \frac{4\epsilon^2}{\sigma(\sqrt{3}k_*)} + 6\right)(u_{01}z_{10} + u_{-1-1}z_{10}) - \left(\frac{2\epsilon^2}{\sigma(0)} + \frac{4\epsilon^2}{\sigma(3k_*)} + 6\right)u_{20}z_{10}$$

$$-\left(\frac{4\epsilon^2}{\sigma(0)} + \frac{4\epsilon^2}{\sigma(\sqrt{3}k_*)} + \frac{4\epsilon^2}{\sigma(\sqrt{7}k_*)} + 6\right)(u_{02}z_{10} + u_{-2-2}z_{10})$$

$$-\left(\frac{8\epsilon^2}{\sigma(\sqrt{3}k_*)} + 6\right)[z_{-10}z_{0-2}z_{22} + z_{20}z_{01}z_{-1-1} + (z_{11}z_{01}z_{0-2} + z_{22}z_{0-1}z_{-1-1})],$$

$$\dot{z}_{20} = \sigma_{20}z_{20} + 2\epsilon z_{22}z_{0-2} + \epsilon z_{10}^2 - \left(\frac{4\epsilon^2}{\sigma(0)} + \frac{4\epsilon^2}{\sigma(3k_*)} + 6\right)u_{10}z_{20}$$

$$-\left(\frac{4\epsilon^2}{\sigma(0)} + \frac{4\epsilon^2}{\sigma(\sqrt{3}k_*)} + \frac{4\epsilon^2}{\sigma(\sqrt{7}k_*)} + 6\right)(u_{01}z_{20} + u_{-1-1}z_{20})$$

$$-\left(\frac{4\epsilon^2}{\sigma(0)} + \frac{2\epsilon^2}{\sigma(4k_*)} + 3\right)u_{20}z_{20} - \left(\frac{4\epsilon^2}{\sigma(0)} + \frac{4\epsilon^2}{\sigma(2\sqrt{3}k_*)} + 6\right)(u_{02}z_{20} + u_{-2-2}z_{20})$$

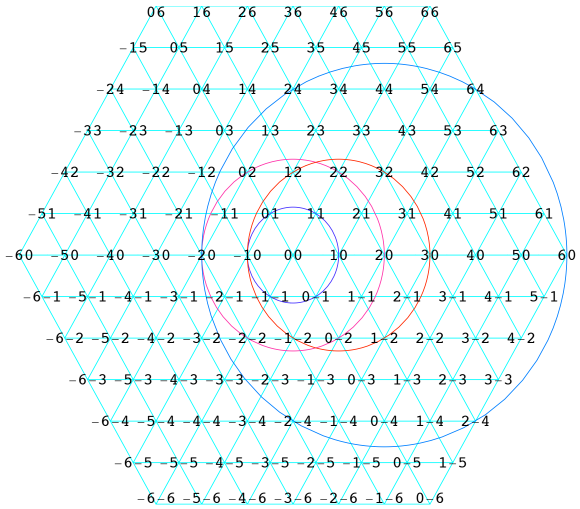
$$-\left(\frac{8\epsilon^2}{\sigma(\sqrt{3}k_*)} + 6\right)z_{10}z_{11}z_{0-1} - \left(\frac{4\epsilon^2}{\sigma(\sqrt{3}k_*)} + 3\right)(z_{22}z_{0-1}^2 + z_{0-2}z_{11}^2).$$

$\star$ $\sigma(k) = r - (1-k^2)^2(4-k^2)^2$

The interaction point locates at $(r_*, k_*) = (0, 1)$.

$$\sigma(0) = -16,\ \sigma(\sqrt{3}) = -4,\ \sigma(\sqrt{7}) = -324,\ \sigma(3) = -1600,$$

$$\sigma(2\sqrt{3}) = -7744,\ \sigma(4) = -32400.$$



Amplitude equations under Z_2-Symmetry

$$\dot{z}_1 = G_1^{(3)}(\boldsymbol{z}) + O(5), \quad \dot{z}_4 = G_4^{(3)}(\boldsymbol{z}) + O(5)$$

$$G_1^{(3)}(\boldsymbol{z}) = \sigma_1 z_1 + [3u_1 + 6(u_2 + u_3)]z_1 + [6u_4 + 6(u_5 + u_6)]z_1$$
$$+6\bar{z}_1\bar{z}_5\bar{z}_6 + 6z_2z_3z_4 + 6(\bar{z}_2z_3\bar{z}_6 + z_2\bar{z}_3\bar{z}_5),$$
$$G_4^{(3)}(\boldsymbol{z}) = \sigma_2 z_4 + [6u_1 + 6(u_2 + u_3)]z_4 + [3u_4 + 6(u_5 + u_6)]z_4$$
$$+6z_1\bar{z}_2\bar{z}_3 + 3(\bar{z}_3^2\bar{z}_5 + \bar{z}_2^2\bar{z}_6).$$

$$u_1 = |z_1|^2,\ u_2 = |z_2|^2,\ u_3 = |z_3|^2,\ u_4 = |z_4|^2,\ u_5 = |z_5|^2,\ u_6 = |z_6|^2$$

At the quintic order approximation,

$$\dot{z}_1 = G_1^{(3)}(\boldsymbol{z}) + \gamma_{3,0}^{1,0,1,0,1,0}u_1^2z_1$$

$$+(\gamma_{1,-2}^{0,-1,0,-1,1,0} + \gamma_{1,2}^{1,0,0,1,0,1})(u_2^2z_1 + u_3^2z_1)$$
$$+\gamma_{5,0}^{1,0,2,0,2,0}u_4^2z_1$$
$$+\gamma_{1,-4}^{0,-2,0,-2,1,0}(u_5^2z_1 + u_6^2z_1)$$
$$+(\gamma_{2,-1}^{0,-1,1,0,1,0} + \gamma_{2,1}^{1,0,1,0,0,1})(u_1u_2z_1 + u_1u_3z_1)$$
$$+(\gamma_{0,0}^{-2,0,1,0,1,0} + \gamma_{4,0}^{1,0,1,0,2,0})u_1u_4z_1$$
$$+\gamma_{2,-2}^{0,-2,1,0,1,0}(u_1u_5z_1 + u_1u_6z_1)$$
$$+\gamma_{0,0}^{1,0,0,1,-1,-1}u_2u_3z_1$$
$$+(\gamma_{-1,1}^{-2,0,1,0,0,1} + \gamma_{3,-1}^{0,-1,1,0,2,0} + \gamma_{3,1}^{1,0,0,1,2,0})(u_2u_4z_1 + u_3u_4z_1)$$
$$+(\gamma_{1,-3}^{0,-2,0,-1,1,0} + \gamma_{1,-1}^{0,-2,1,0,0,1} + \gamma_{1,3}^{1,0,0,1,0,2})(u_2u_5z_1 + u_3u_6z_1)$$
$$+(\gamma_{3,1}^{2,2,0,-1,1,0} + \gamma_{3,3}^{2,2,1,0,0,1} + \gamma_{-1,-3}^{0,-1,1,0,-2,-2})(u_2u_6z_1 + u_3u_5z_1)$$
$$+(\gamma_{3,2}^{1,0,2,0,0,2} + \gamma_{-1,-2}^{0,-2,-2,0,1,0} + \gamma_{3,-2}^{0,-2,1,0,2,0})(u_4u_5z_1 + u_4u_6z_1)$$
$$+(\gamma_{3,4}^{2,2,1,0,0,2} + \gamma_{3,0}^{2,2,0,-2,1,0} + \gamma_{-1,-4}^{0,-2,1,0,-2,-2})u_5u_6z_1$$
$$+\gamma_{3,0}^{2,2,0,-2,1,0}u_1\bar{z}_1\bar{z}_5\bar{z}_6$$

$$+(\gamma_{2,-1}^{0,-2,1,1,1,0} + \gamma_{1,-1}^{0,-2,1,0,0,1})(u_1z_2\bar{z}_3\bar{z}_5 + u_1\bar{z}_2z_3\bar{z}_6)$$
$$+(\gamma_{3,1}^{1,0,0,1,2,0} + \gamma_{2,-1}^{1,0,-1,-1,2,0} + \gamma_{0,0}^{1,0,0,1,-1,-1})u_1z_2z_3z_4$$
$$+(\gamma_{-1,1}^{0,1,0,1,-1,-1} + \gamma_{1,-2}^{0,-1,-1,-1,2,0})(u_2z_2z_3z_4 + u_3z_2z_3z_4)$$
$$+(\gamma_{-1,-3}^{0,-1,0,-1,-1,-1} + \gamma_{1,2}^{2,2,0,1,-1,-1})(u_2\bar{z}_2z_3\bar{z}_6 + u_3z_2\bar{z}_3\bar{z}_5)$$
$$+(\gamma_{1,-2}^{0,-2,1,1,0,-1} + \gamma_{1,3}^{1,1,0,1,0,1} + \gamma_{0,0}^{0,-2,0,1,0,1})(u_2z_2\bar{z}_3\bar{z}_5 + u_3\bar{z}_2z_3\bar{z}_6)$$
$$+(\gamma_{2,-1}^{2,2,0,-2,0,-1} + \gamma_{2,1}^{2,2,0,-2,0,1} + \gamma_{1,3}^{2,2,-1,0,0,1} + \gamma_{-1,-3}^{0,-2,0,-1,-1,0})(u_2\bar{z}_1\bar{z}_5\bar{z}_6 + u_3\bar{z}_1\bar{z}_5\bar{z}_6)$$
$$+(\gamma_{4,1}^{0,1,2,0,2,0} + \gamma_{3,-1}^{-1,-1,2,0,2,0})u_4z_2z_3z_4$$
$$+(\gamma_{0,0}^{2,2,0,-2,-2,0} + \gamma_{4,0}^{2,2,0,-2,2,0} + \gamma_{3,2}^{2,2,-1,0,2,0} + \gamma_{1,-2}^{0,-2,-1,0,2,0})u_4\bar{z}_1\bar{z}_5\bar{z}_6$$
$$+(\gamma_{-2,-1}^{0,-2,-2,0,0,1} + \gamma_{3,2}^{1,1,0,1,2,0} + \gamma_{3,-1}^{0,-2,1,1,2,0} + \gamma_{2,-1}^{0,-2,0,1,2,0})(u_4z_2\bar{z}_3\bar{z}_5 + u_4\bar{z}_2z_3\bar{z}_6)$$
$$+(\gamma_{-1,-2}^{0,-2,0,1,-1,-1} + \gamma_{2,3}^{0,1,2,0,0,2} + \gamma_{2,-1}^{0,-2,0,1,2,0} + \gamma_{1,-3}^{0,-2,-1,-1,2,0})(u_5z_2z_3z_4 + u_6z_2z_3z_4)$$
$$+(\gamma_{1,-3}^{0,-2,0,-2,1,1} + \gamma_{0,-3}^{0,-2,0,-2,0,1})(u_5z_2\bar{z}_3\bar{z}_5 + u_6\bar{z}_2z_3\bar{z}_6)$$
$$+(\gamma_{3,1}^{2,2,0,-2,1,1} + \gamma_{2,1}^{2,2,0,-2,0,1} + \gamma_{3,4}^{2,2,1,1,0,1} + \gamma_{-1,-3}^{0,-2,1,1,-2,-2} + \gamma_{-2,-3}^{0,-2,0,1,-2,-2})(u_6z_2\bar{z}_3\bar{z}_5 + u_5\bar{z}_2z_3\bar{z}_6)$$
$$+(\gamma_{4,2}^{2,2,2,2,0,-2} + \gamma_{3,4}^{2,2,2,2,-1,0})(u_6\bar{z}_1\bar{z}_5\bar{z}_6 + u_5\bar{z}_1\bar{z}_5\bar{z}_6)$$
$$+(\gamma_{2,1}^{1,1,1,1,0,-1} + \gamma_{0,0}^{1,1,0,-1,-1,0} + \gamma_{-1,-2}^{0,-1,0,-1,-1,0} + \gamma_{1,-1}^{1,1,0,-1,0,-1} + \gamma_{1,2}^{1,1,1,1,-1,0})\bar{z}_1\bar{z}_2^2\bar{z}_3^2$$
$$+(\gamma_{0,0}^{-1,0,-1,0,2,0} + \gamma_{3,0}^{-1,0,2,0,2,0})\bar{z}_1^3z_4^2$$
$$+\gamma_{3,0}^{1,0,1,0,1,0}z_1^3z_5z_6$$

$$+(\gamma^{0,-1,1,0,1,0}_{2,-1}+\gamma^{-2,0,1,0,1,0}_{0,0}+\gamma^{1,1,1,0,1,0}_{3,1})z_1^2\bar z_2\bar z_3\bar z_4$$
$$+(\gamma^{0,-1,-1,0,2,0}_{1,-1}+\gamma^{-1,0,-1,0,2,0}_{0,0}+\gamma^{1,1,0,-1,-1,0}_{0,0}+\gamma^{1,1,-1,0,2,0}_{2,1}+\gamma^{1,1,-1,0,-1,0}_{-1,1}+\gamma^{0,-1,-1,0,-1,0}_{-2,-1}+\gamma^{1,1,0,-1,2,0}_{3,0})\bar z_1^2\bar z_2\bar z_3 z_4$$
$$+(\gamma^{2,0,2,0,-2,-2}_{2,-2}+\gamma^{2,0,0,2,-2,-2}_{0,0}+\gamma^{-1,0,2,0,2,0}_{3,0}+\gamma^{2,0,2,0,0,2}_{4,2}+\gamma^{-1,0,2,0,0,2}_{1,2}+\gamma^{-1,0,2,0,-2,-2}_{-1,-2})\bar z_1 z_4^2 z_5 z_6$$
$$+(\gamma^{2,2,0,-2,-2,0}_{0,0}+\gamma^{2,2,0,-2,1,1}_{3,1}+\gamma^{2,2,-2,0,1,1}_{1,3}+\gamma^{2,2,0,-2,0,-1}_{2,-1}+\gamma^{2,2,1,1,0,-1}_{3,2}+\gamma^{0,-2,-2,0,0,-1}_{-2,-3}+\gamma^{0,-2,1,1,0,-1}_{1,-2})\bar z_2\bar z_3\bar z_4\bar z_5\bar z_6$$
$$+(\gamma^{1,1,0,-1,2,0}_{3,0}+\gamma^{1,1,2,0,-2,-2}_{1,-1}+\gamma^{1,1,0,-1,0,2}_{1,2}+\gamma^{1,1,0,2,-2,-2}_{-1,1}+\gamma^{0,-1,2,0,0,2}_{2,1}+\gamma^{2,0,0,2,-2,-2}_{0,0}+\gamma^{0,-1,0,2,-2,-2}_{-2,-1}$$
$$+\gamma^{0,-1,2,0,-2,-2}_{0,-3}+\gamma^{1,1,0,-1,-2,-2}_{-1,-2}+\gamma^{1,1,2,0,0,2}_{3,3})\bar z_2\bar z_3 z_4 z_5 z_6$$
$$+(\gamma^{0,-2,-1,0,2,0}_{1,-2}+\gamma^{0,-2,0,1,0,1}_{0,0}+\gamma^{0,-2,0,1,2,0}_{2,-1})(\bar z_1 z_2^2 z_4\bar z_5+\bar z_1 z_3^2 z_4\bar z_6)$$
$$+(\gamma^{2,2,-2,0,1,0}_{1,2}+\gamma^{0,-1,0,-1,1,0}_{1,-2}+\gamma^{2,2,0,-1,1,0}_{3,1})(z_1\bar z_2^2\bar z_4\bar z_6+z_1\bar z_3^2\bar z_4\bar z_5)$$
$$+(\gamma^{1,1,0,-1,0,-1}_{1,-1}+\gamma^{0,-1,0,-1,0,2}_{0,0}+\gamma^{1,1,0,-1,0,2}_{1,2}+\gamma^{0,-1,0,-1,0,-1}_{0,-3})(\bar z_2^3\bar z_3 z_5+\bar z_2\bar z_3^3 z_6)$$
$$+(\gamma^{0,-1,1,0,1,0}_{2,-1}+\gamma^{1,0,1,0,-1,-1}_{1,-1})(z_1^2\bar z_2 z_3 z_5+z_1^2 z_2\bar z_3 z_6)$$
$$+(\gamma^{0,-1,0,-1,-1,0}_{-1,-2}+\gamma^{0,-1,0,-1,2,0}_{2,-2}+\gamma^{0,-1,0,-1,0,2}_{0,0}+\gamma^{0,-1,-1,0,2,0}_{1,-1}+\gamma^{0,-1,-1,0,0,2}_{-1,1}+\gamma^{0,-1,2,0,0,2}_{2,1}+\gamma^{-1,0,2,0,0,2}_{1,2})$$
$$\times(\bar z_1\bar z_2^2 z_4 z_5+\bar z_1\bar z_3^2 z_4 z_6)$$
$$+(\gamma^{1,0,2,0,-2,-2}_{1,-2}+\gamma^{1,0,0,1,0,1}_{1,2}+\gamma^{1,0,0,1,2,0}_{3,1})(z_1 z_2^2 z_4 z_6+z_1 z_3^2 z_4 z_5)$$

$$\dot z_4 = G_4^{(3)}(\boldsymbol z)+(\gamma^{-1,0,-1,0,2,0}_{0,0}+\gamma^{1,0,1,0,2,0}_{4,0})u_1^2 z_4$$
$$+\gamma^{0,-1,0,-1,2,0}_{2,-2}(u_2^2 z_4+u_3^2 z_4)$$
$$+\gamma^{2,0,2,0,2,0}_{6,0}u_4^2 z_4$$
$$+(\gamma^{0,-2,0,-2,2,0}_{2,-4}+\gamma^{2,0,0,2,0,2}_{2,4})(u_5^2 z_4+u_6^2 z_4)$$
$$+(\gamma^{0,-1,-1,0,2,0}_{1,-1}+\gamma^{0,-1,1,0,2,0}_{3,-1}+\gamma^{1,0,0,1,2,0}_{3,1})(u_1u_2z_4+u_1u_3z_4)$$
$$+(\gamma^{-1,0,2,0,2,0}_{3,0}+\gamma^{1,0,2,0,2,0}_{5,0})u_1u_4z_4$$
$$+(\gamma^{0,-2,-1,0,2,0}_{1,-2}+\gamma^{0,-2,1,0,2,0}_{3,-2}+\gamma^{-1,0,2,0,0,2}_{1,2}+\gamma^{1,0,2,0,0,2}_{3,2})(u_1u_5z_4+u_1u_6z_4)$$
$$+(\gamma^{0,-2,0,-1,2,0}_{2,-3}+\gamma^{0,-2,0,1,2,0}_{2,-1}+\gamma^{0,-1,2,0,0,2}_{2,1}+\gamma^{0,1,2,0,0,2}_{2,3})(u_2u_5z_4+u_3u_6z_4)$$
$$+(\gamma^{1,1,0,-1,2,0}_{3,0}+\gamma^{1,1,0,1,2,0}_{3,2}+\gamma^{0,-1,-1,-1,2,0}_{1,-2})u_2u_3z_4$$
$$+(\gamma^{0,-1,2,0,2,0}_{4,-1}+\gamma^{0,1,2,0,2,0}_{4,1})(u_2u_4z_4+u_3u_4z_4)$$
$$+(\gamma^{2,2,0,-1,2,0}_{4,1}+\gamma^{2,2,0,1,2,0}_{4,3}+\gamma^{0,-1,2,0,-2,-2}_{0,-3})(u_2u_6z_4+u_3u_5z_4)$$
$$+(\gamma^{0,-2,2,0,2,0}_{4,-2}+\gamma^{2,0,2,0,0,2}_{4,2})(u_4u_5z_4+u_4u_6z_4)$$
$$+(\gamma^{2,2,0,-2,2,0}_{4,0}+\gamma^{2,2,2,0,0,2}_{4,4}+\gamma^{0,-2,2,0,-2,-2}_{0,-4}+\gamma^{2,0,0,2,-2,-2}_{0,0})u_5u_6z_4$$
$$+(\gamma^{1,1,0,-1,-1,0}_{0,0}+\gamma^{1,1,1,0,1,0}_{3,1})+\gamma^{0,-1,1,0,1,0}_{2,-1})u_1z_1\bar z_2\bar z_3$$
$$+(\gamma^{2,2,0,-1,1,0}_{3,1}+\gamma^{0,-1,0,-1,-1,0}_{-1,-2}+\gamma^{0,-1,0,-1,1,0}_{1,-2})(u_1\bar z_2^2\bar z_6+u_1\bar z_3^2\bar z_5)$$

$$+(\gamma^{0,-2,1,1,0,-1}_{1,-2}+\gamma^{1,1,1,1,0,-1}_{2,1}+\gamma^{1,1,1,1,0,1}_{2,3})(u_2\bar z_3^2\bar z_5+u_3\bar z_2^2\bar z_6)$$
$$+(\gamma^{1,1,0,-1,0,-1}_{1,-1}+\gamma^{0,-1,0,-1,1,0}_{1,-2})(u_2z_1\bar z_2\bar z_3+u_3z_1\bar z_2\bar z_3)$$
$$+\gamma^{0,-1,0,-1,0,-1}_{0,-3}(u_2\bar z_2^2\bar z_6+u_3\bar z_3^2\bar z_5)$$
$$+(\gamma^{1,1,0,-1,2,0}_{3,0}+\gamma^{1,1,1,0,2,0}_{4,1}+\gamma^{0,-1,1,0,2,0}_{3,-1})u_4z_1\bar z_2\bar z_3$$
$$+(\gamma^{0,-1,0,-1,2,0}_{2,-2}+\gamma^{2,2,0,-1,2,0}_{4,1})(u_4\bar z_2^2\bar z_6+u_4\bar z_3^2\bar z_5)$$
$$+(\gamma^{2,2,0,-2,0,-1}_{2,-1}+\gamma^{2,2,0,-1,0,2}_{2,3}+\gamma^{0,-2,0,-1,0,-1}_{0,-4}+\gamma^{0,-1,0,-1,0,2}_{0,0})(u_5\bar z_2^2\bar z_6+u_6\bar z_3^2\bar z_5)$$
$$+(\gamma^{0,-2,0,-2,1,1}_{1,-3}+\gamma^{1,1,1,1,0,2}_{2,4})(u_5\bar z_3^2\bar z_5+u_6\bar z_2^2\bar z_6)$$
$$+(\gamma^{0,-2,1,1,0,-1}_{1,-2}+\gamma^{0,-2,1,1,1,0}_{2,-1}+\gamma^{0,-2,0,-1,1,0}_{1,-3}+\gamma^{1,1,0,-1,0,2}_{1,2}+\gamma^{1,1,1,0,0,2}_{2,3})(u_5z_1\bar z_2\bar z_3+u_6z_1\bar z_2\bar z_3)$$
$$+(\gamma^{-2,0,1,0,1,0}_{0,0}+\gamma^{1,0,1,0,1,0}_{3,0})z_1^4\bar z_4$$
$$+(\gamma^{0,1,2,0,2,0}_{4,1}+\gamma^{2,0,2,0,-2,-2}_{2,-2})(z_2^2z_4^2z_6+z_3^2z_4^2z_5)$$
$$+(\gamma^{2,2,2,2,0,-2}_{4,2}+\gamma^{2,2,2,2,-2,0}_{2,4}+\gamma^{2,2,0,-2,0,-2}_{2,-2}+\gamma^{2,2,0,-2,-2,0}_{0,0}+\gamma^{0,-2,0,-2,-2,0}_{-2,-4})\bar z_4\bar z_5^2\bar z_6^2$$
$$+(\gamma^{2,2,2,2,0,-2}_{4,2}+\gamma^{2,2,2,2,-1,-1}_{3,3}+\gamma^{2,2,0,-2,-1,-1}_{1,-1}+\gamma^{2,2,-1,-1,-1,-1}_{0,0}+\gamma^{0,-2,-1,-1,-1,-1}_{-2,-4})(z_3^2\bar z_5\bar z_6^2+z_2^2\bar z_5^2\bar z_6)$$
$$+(\gamma^{1,0,1,0,2,0}_{4,0}+\gamma^{1,0,2,0,0,2}_{3,2}+\gamma^{1,0,2,0,-2,-2}_{1,-2}+\gamma^{2,0,0,2,-2,-2}_{0,0})z_1^2z_4z_5z_6$$
$$+(\gamma^{2,2,0,-2,-2,0}_{0,0}+\gamma^{2,2,0,-2,1,0}_{3,0}+\gamma^{2,2,-2,0,1,0}_{1,2}+\gamma^{2,2,1,0,1,0}_{4,2}+\gamma^{0,-2,-2,0,1,0}_{-1,-2}+\gamma^{0,-2,1,0,1,0}_{2,-2}+\gamma^{-2,0,1,0,1,0}_{0,0})z_1^2\bar z_4\bar z_5\bar z_6$$
$$+(\gamma^{2,2,0,-2,2,0}_{4,0}+\gamma^{2,2,-1,0,2,0}_{3,2}+\gamma^{0,-2,-1,0,2,0}_{1,-2}+\gamma^{-1,0,-1,0,2,0}_{0,0})\bar z_1^2z_4\bar z_5\bar z_6$$
$$+(\gamma^{2,2,0,-2,1,0}_{3,0}+\gamma^{2,2,0,-2,0,1}_{2,1}+\gamma^{2,2,0,-2,-1,-1}_{1,-1}+\gamma^{2,2,1,0,0,1}_{3,3}+\gamma^{2,2,1,0,-1,-1}_{2,1}+\gamma^{2,2,0,1,-1,-1}_{1,2}+\gamma^{0,-2,1,0,0,1}_{1,-1}$$

$$+\gamma^{0,-2,1,0,-1,-1}_{0,-3}+\gamma^{0,-2,0,1,-1,-1}_{-1,-2}+\gamma^{1,0,0,1,-1,-1}_{0,0})z_1z_2z_3\bar z_5\bar z_6$$
$$+(\gamma^{2,2,0,-2,1,1}_{3,1}+\gamma^{2,2,0,-2,0,-1}_{2,-1}+\gamma^{2,2,1,1,0,-1}_{3,2}+\gamma^{2,2,1,1,-1,0}_{2,3}+\gamma^{0,-2,1,1,0,-1}_{1,-2}+\gamma^{0,-2,0,-1,-1,0}_{-1,-3}+\gamma^{1,1,0,-1,-1,0}_{0,0})\bar z_1\bar z_2\bar z_3\bar z_5\bar z_6$$
$$+(\gamma^{0,-2,1,1,2,0}_{3,-1}+\gamma^{0,-2,-1,0,2,0}_{1,-2}+\gamma^{0,-2,0,1,2,0}_{2,-1}+\gamma^{1,1,-1,0,2,0}_{2,1}+\gamma^{1,1,0,1,2,0}_{3,2})(\bar z_1z_2\bar z_3z_4\bar z_5+\bar z_1\bar z_2z_3z_4\bar z_6)$$
$$+(\gamma^{0,-2,1,0,1,0}_{2,-2}+\gamma^{0,-2,1,0,0,1}_{1,-1}+\gamma^{0,-2,0,1,0,1}_{0,0}+\gamma^{1,0,1,0,0,1}_{2,1}+\gamma^{1,0,0,1,0,1}_{1,2})(z_1^2z_2^2\bar z_5+z_1^2z_3^2\bar z_6)$$
$$+(\gamma^{0,-1,0,-1,1,0}_{1,-2}+\gamma^{0,-1,0,-1,0,2}_{0,0}+\gamma^{0,-1,1,0,1,0}_{2,-1})(z_1^2\bar z_2^2z_5+z_1^2\bar z_3^2z_6)$$
$$+(\gamma^{1,1,1,0,2,0}_{4,1}+\gamma^{1,1,0,1,2,0}_{3,2}+\gamma^{1,1,2,0,-2,-2}_{1,-1}+\gamma^{1,0,0,1,2,0}_{3,1}+\gamma^{1,0,2,0,-2,-2}_{1,-2})(z_1z_2\bar z_3z_4z_6+z_1\bar z_2z_3z_4z_5)$$
$$+(\gamma^{-1,0,2,0,2,0}_{3,0}+\gamma^{0,1,2,0,2,0}_{4,1}+\gamma^{-1,-1,2,0,2,0}_{3,-1})\bar z_1z_2z_3z_4^2$$
$$+(\gamma^{1,0,1,0,1,0}_{3,0}+\gamma^{1,0,1,0,0,1}_{2,1}+\gamma^{1,0,1,0,-1,-1}_{1,-1}+\gamma^{1,0,0,1,-1,-1}_{0,0})z_1^3z_2z_3$$

Invariant subspaces in the absence of Z_2

$I_1 = \mathbb{C}(0,0,0,1,0,0)$

$I_2 = \mathbb{C}\{(1,0,0,0,0,0),(0,0,0,1,0,0)\}$

$I_3 = \mathbb{C}\{(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\}$

$I_4 = \mathbb{C}\{(1,0,0,0,0,0),(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\}$

$$I_6 = \mathbb{C}\{(1,0,0,0,0,0),(0,1,0,0,0,0),(0,0,1,0,0,0),\\ (0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\}$$

Invariant subspaces in the presence of Z_2

$I_1 = \mathbb{C}(0,0,0,1,0,0)$

$I_1' = \mathbb{C}(1,0,0,0,0,0)$

$I_2 = \mathbb{C}\{(1,0,0,0,0,0),(0,0,0,1,0,0)\}$

$I_2' = \mathbb{C}\{(0,1,0,0,0,0),(0,0,1,0,0,0)\}$

$I_2'' = \mathbb{C}\{(0,0,0,0,1,0),(0,0,0,0,0,1)\}$

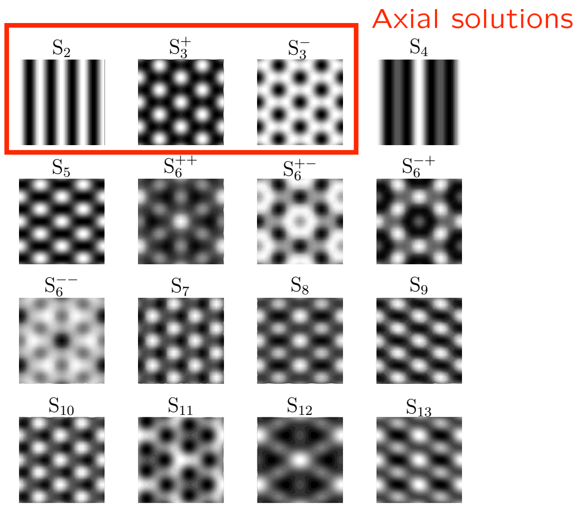
$I_3 = \mathbb{C}\{(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\}$

$I_3' = \mathbb{C}\{(1,0,0,0,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\}$

$I_4 = \mathbb{C}\{(1,0,0,0,0,0),(0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\}$

$$I_6 = \mathbb{C}\{(1,0,0,0,0,0),(0,1,0,0,0,0),(0,0,1,0,0,0),\\ (0,0,0,1,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\}$$

Steady planform without Z_2



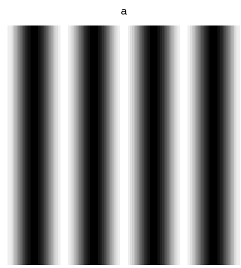
Steady Solutions

Trivial solution: (0,0,0,0,0,0) $D_6 \dotplus T^2$
$\{R_{2\pi/3}, c, c_v, S^1(\theta,0), S^1(0,\theta)\}$

Pure mode (Rolls): $(0,0,0,x,0,0), x \in \mathbb{R}$ $S^1 + Z_2^3$
$\{c, c_v, Z_2(\pi,0), S^1(0,\theta)\}$ $\mathbb{R}\{(0,0,0,1,0,0)\}$

SR

$$\sigma_2 + \mu_{21} x^2 = 0$$

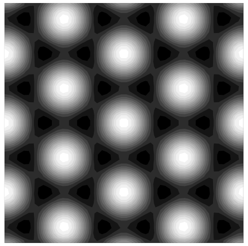


Pure mode (Hexagons): $(0,0,0,x,x,x)$, $x \in \mathbb{R}$ $D_6 + Z_2^2$
$\{R_{2\pi/3}, c, c_v, Z_2(\pi,0), Z_2(0,\pi)\}$ $\mathbb{R}\{(0,0,0,1,1,1)\}$

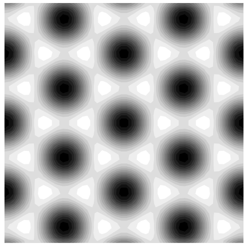
SH

$$\sigma_2 + \delta_2 x + (\mu_{21} + 2\mu_{22})x^2 = 0$$

b



c



up/down hexagons

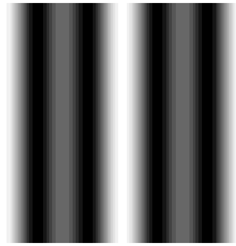
Mixed mode (M-Rolls): $(x,0,0,y,0,0)$, $x, y \in \mathbb{R}$ $S^1 + Z_2^2$
$\{c, c_v, S^1(0,\theta)\}$ $\mathbb{R}\{(1,0,0,0,0,0),(0,0,0,1,0,0)\}$

MR

$$\sigma_1 + \beta_1 y + \kappa_{11}x^2 + \mu_{11}y^2 = 0,$$

$$\sigma_2 y + \beta_2 x^2 + \kappa_{21}x^2 y + \mu_{21}y^3 = 0$$

d



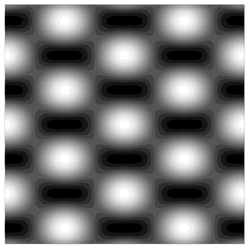
Pure mode (Rectangles): $(0,0,0,x,y,y)$, $x \neq y \in \mathbb{R}$ Z_2^4
$\{c, c_v, Z_2(\pi,0), Z_2(0,\pi)\}$ $\mathbb{R}\{(0,0,0,1,0,0),(0,0,0,0,1,1)\}$

RA

$$\sigma_2 x + \delta_2 y^2 + (\mu_{21}x^2 + 2\mu_{22}y^2)x = 0,$$

$$\sigma_2 + \delta_2 x + \mu_{22}x^2 + (\mu_{21} + \mu_{22})y^2 = 0$$

e



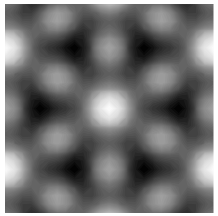
Mixed mode (M-Hexagons): (x,x,x,y,y,y), $x,y \in \mathbb{R}$ D_6
$\{R_{2\pi/3}, c, c_v\}$ $\mathbb{R}\{(1,1,1,0,0,0),(0,0,0,1,1,1)\}$

MH

$$\sigma_1 + \beta_1 y + \delta_1 x + (\kappa_{11} + 2\kappa_{12})x^2 + (\mu_{11} + 2\mu_{12} + \nu_1)y^2 + (2\eta_1 + \xi_1)xy = 0,$$

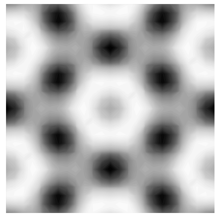
$$\sigma_2 y + \beta_2 x^2 + \delta_2 y^2 + (\kappa_{21} + 2\kappa_{22} + 2\xi_2)x^2 y + (\mu_{21} + 2\mu_{22})y^3 + \nu_2 x^3 = 0$$

f



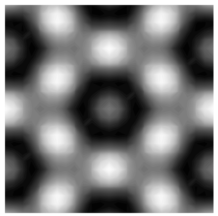
++

g



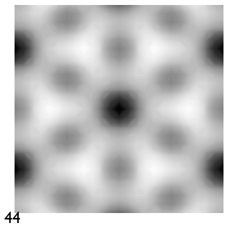
+−

h



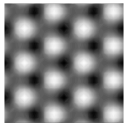
−+

h

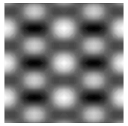


−−

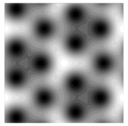
Pure mode (Triangles): $(0,0,0,z,z,z)$, $z\in\mathbb{C}$ $D_3+Z_2^2$
$\{R_{2\pi/3}, c_v, Z_2(\pi,0), Z_2(0,\pi)\}$ $\mathbb{C}\{(0,0,0,1,1,1)\}$



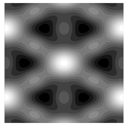
Mixed mode (Rect.Rolls): $(z,0,0,x,y,y)$, $z,x\neq y\in\mathbb{R}$ Z_2^3
$\{c, c_v, Z_2(0,\pi)\}$ $\mathbb{R}\{(1,0,0,0,0,0),(0,0,0,1,0,0),(0,0,0,0,1,1)\}$
R-RA



Mixed mode (M-Triangles): (x,x,x,y,y,y), $x,y\in\mathbb{C}$ D_3
$\{R_{2\pi/3}, c_v\}$ $\mathbf{C}\{(1,1,1,0,0,0),(0,0,0,1,1,1)\}$



Mixed mode (M-Rectangles): (x,y,y,u,v,v), $x\neq y, u\neq v\in\mathbb{R}$ Z_2^2
$\{c, c_v\}$ $\mathbb{R}\{(1,0,0,0,0,0),(0,1,1,0,0,0),(0,0,0,1,0,0),(0,0,0,0,1,1)\}$



Under Z_2-symmetry

Large rolls: $(x,0,0,0,0,0)$, $x\in\mathbb{R}$.

Small rolls: $(0,0,0,x,0,0)$, $x\in\mathbb{R}$.

Large patchwork quilts: $(0,x,x,0,0,0)$, $x\in\mathbb{R}$.

Small patchwork quilts: $(0,0,0,0,x,x)$, $x\in\mathbb{R}$.

Small hexagons: $(0,0,0,x,x,x)$, $x\in\mathbb{R}$.

Small triangles: $(0,0,0,ix,ix,ix)$, $z\in\mathbb{R}$.

Axial solutions

Rectangles: $(0,0,0,x,y,y)$, $x,y\in\mathbb{R}$.

Mixed rolls: $(x,0,0,y,0,0)$, $x,y\in\mathbb{R}$.

Roll-patchwork quilts: $(x,0,0,0,y,y)$, $x,y\in\mathbb{R}$.

Mixed hexagons: (x,x,x,y,y,y), $x,y\in\mathbb{R}$.

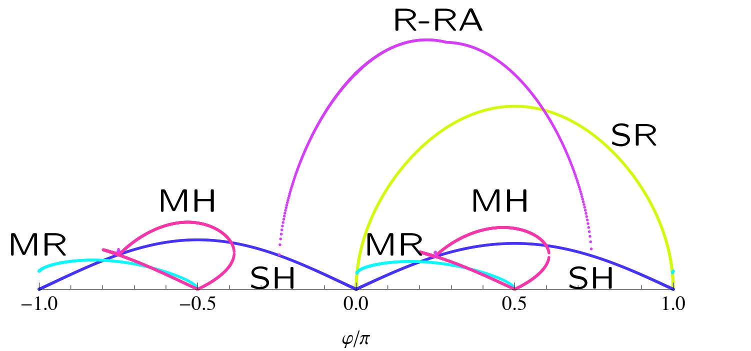
Mixed triangles: (ix,ix,ix,iy,iy,iy), $x,y\in\mathbb{R}$.

Roll-rectangles: $(x,0,0,y,z,z)$, $x,y,z\in\mathbb{R}$.

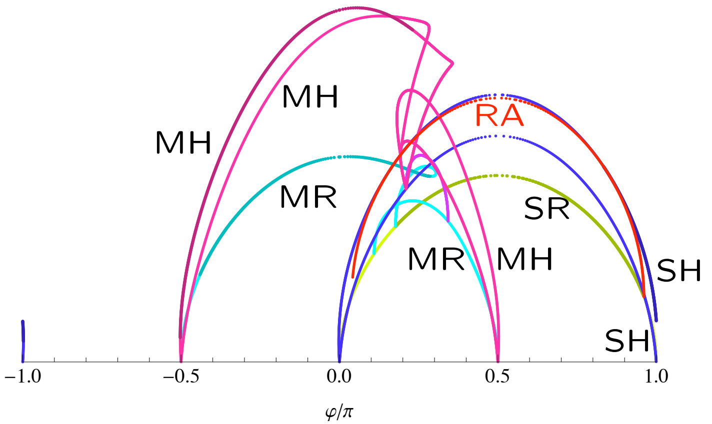
$$
\begin{aligned}
\dot z_{10} = {} & \sigma_{10} z_{10} + 2\epsilon z_{11} z_{0-1} + 2\epsilon z_{20} z_{-10} - \left(\frac{4\epsilon^2}{\sigma(0)} + 3\right) u_{10} z_{10} \\
& - \left(\frac{4\epsilon^2}{\sigma(0)} + \frac{4\epsilon^2}{\sigma(\sqrt{3}k_*)} + 6\right)(u_{01} z_{10} + u_{-1-1} z_{10}) - \left(\frac{2\epsilon^2}{\sigma(0)} + \frac{4\epsilon^2}{\sigma(3k_*)} + 6\right) u_{20} z_{10} \\
& - \left(\frac{4\epsilon^2}{\sigma(0)} + \frac{4\epsilon^2}{\sigma(\sqrt{3}k_*)} + \frac{4\epsilon^2}{\sigma(\sqrt{7}k_*)} + 6\right)(u_{02} z_{10} + u_{-2-2} z_{10}) \\
& - \left(\frac{8\epsilon^2}{\sigma(\sqrt{3}k_*)} + 6\right)[z_{-10} z_{0-2} z_{22} + z_{20} z_{01} z_{-1-1} + (z_{11} z_{01} z_{0-2} + z_{22} z_{0-1} z_{-1-1})], \\
\dot z_{20} = {} & \sigma_{20} z_{20} + 2\epsilon z_{22} z_{0-2} + \epsilon z_{10}^2 - \left(\frac{4\epsilon^2}{\sigma(0)} + \frac{4\epsilon^2}{\sigma(3k_*)} + 6\right) u_{10} z_{20} \\
& - \left(\frac{4\epsilon^2}{\sigma(0)} + \frac{4\epsilon^2}{\sigma(\sqrt{3}k_*)} + \frac{4\epsilon^2}{\sigma(\sqrt{7}k_*)} + 6\right)(u_{01} z_{20} + u_{-1-1} z_{20}) \\
& - \left(\frac{4\epsilon^2}{\sigma(0)} + \frac{2\epsilon^2}{\sigma(4k_*)} + 3\right) u_{20} z_{20} - \left(\frac{4\epsilon^2}{\sigma(0)} + \frac{4\epsilon^2}{\sigma(2\sqrt{3}k_*)} + 6\right)(u_{02} z_{20} + u_{-2-2} z_{20}) \\
& - \left(\frac{8\epsilon^2}{\sigma(\sqrt{3}k_*)} + 6\right) z_{10} z_{11} z_{0-1} - \left(\frac{4\epsilon^2}{\sigma(\sqrt{3}k_*)} + 3\right)(z_{22} z_{0-1}^2 + z_{0-2} z_{11}^2).
\end{aligned}
$$

$\sigma_{10} = r\cos\varphi$, $\sigma_{20} = r\sin\varphi$

SH,MR,MH: stable

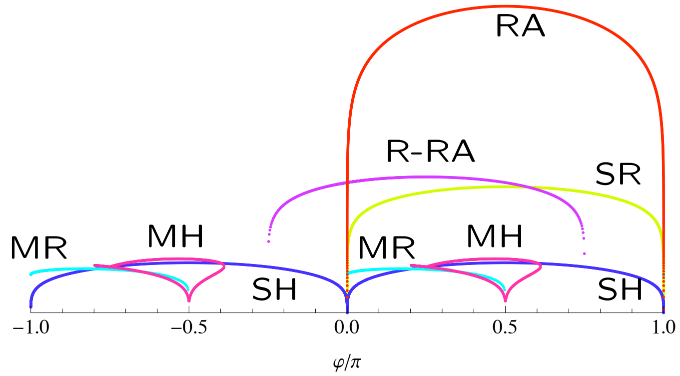


$\epsilon = 0.1$, $r = 10^{-4}$ $\cdots + \epsilon u^2$

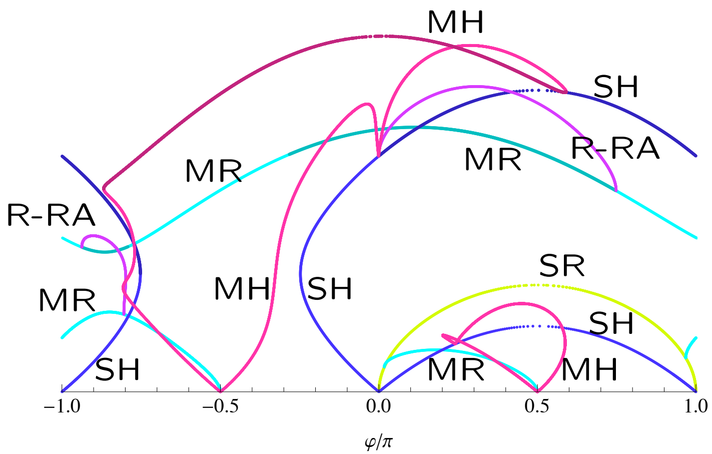


$\epsilon = 0.1, \ r = 10^{-1}$

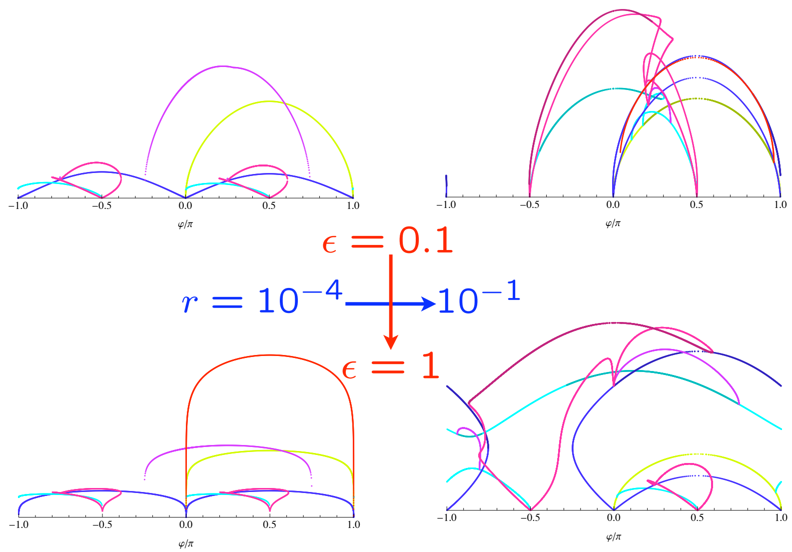
SH,MR,MH: stable

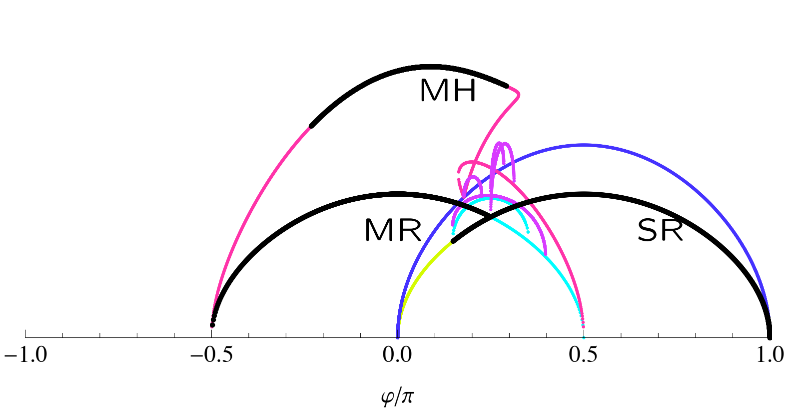


$\epsilon = 1, \ r = 10^{-4}$ $\qquad \cdots + \epsilon u^2$

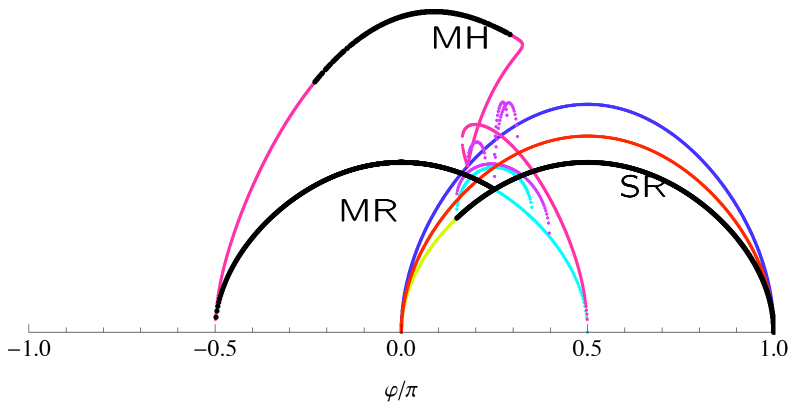


$\epsilon = 1, \ r = 10^{-1}$

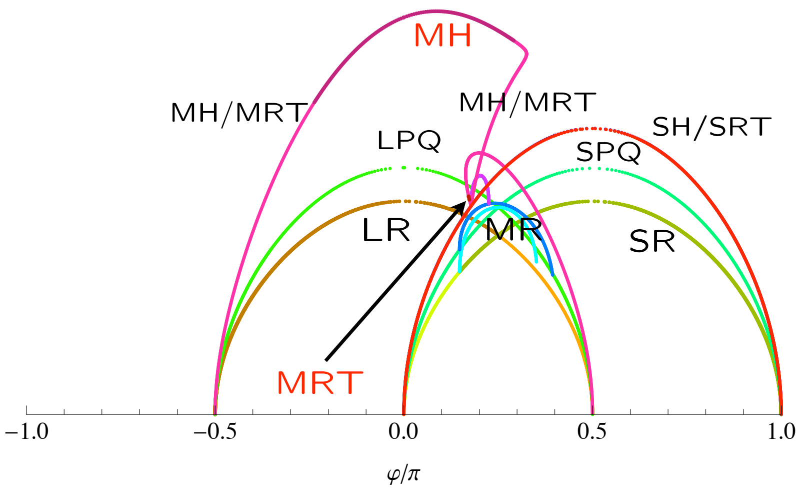




$\epsilon = 0, \;\; r = 10^{-4}$, cubic



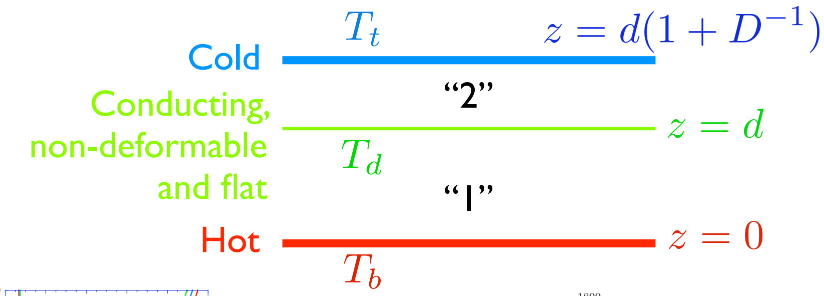
$\epsilon = 0, \;\; r = 10^{-1}$, cubic

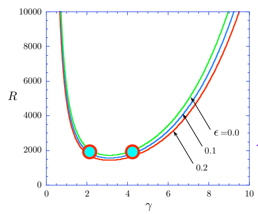


$\epsilon = 0, \;\; r = 10^{-1}$, quintic

Physical Setup -- Two-layered RB Problem

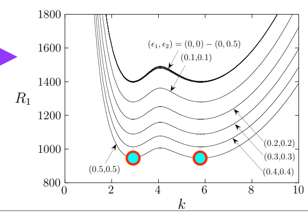
Proctor & Jones, J.Fluid Mech. **188** *(1988) pp.301–355.*





single-layer

two-layer



1. Two-layered Rayleigh-Bénard problem

$$\rho_0^{(j)} \frac{D\vec{v}_j^*}{Dt^*} = -\nabla^* p_j^* - \rho_0^{(j)} g[1 - \alpha_1^{(j)}(T_j^* - T_d) - \alpha_2^{(j)}(T_j^* - T_d)^2]\mathbf{e}_z + \mu_j \Delta^* \vec{v}_j^*,$$

$$\frac{DT_j^*}{Dt^*} = \kappa_j \Delta^* T_j^*, \quad \nabla^* \cdot \vec{v}_j^* = 0, \quad (j = 1, 2). \tag{1.1}$$

$\mathbf{e}_z$: the unit vector upward in the z-direction,
g: the acceleration due to the gravity,
μ_1, μ_2: the viscous coefficients,
κ_1, κ_2: the thermal diffusivitpares,
$\rho_0^{(1)}$, $\rho_0^{(2)}$: the densities.
physical properties: evaluated at $T_1^* = T_d$ and $T_2^* = T_d$.

2. Non-dimensionalization

$$t^* = \frac{d^2}{\kappa_1} t, \quad \vec{v}_j^* = \frac{\kappa_j}{d} \vec{v}_j, \quad \vec{x}^* = d\vec{x},$$

$$p_j^* = -d\rho_0^{(j)} g \int^z [1 - \alpha_1^{(1)} \tilde{D}_j (T^{(j)} - T_d)(1-z) - \alpha_2^{(j)} \tilde{D}_j^2 (T^{(j)} - T_d)^2 (1-z)^2] dz + \rho_0^{(j)} \frac{\kappa_j^2}{d^2} \pi_1,$$

$$T_j^* - T_d = (T^{(j)} - T_d)\tilde{D}_j[(1 - z) + \theta_j(x, y, z; t)], \ (j = 1, 2) \tag{2.1}$$

where we set $\tilde{D}_j = (-D)^{j-1}$, $T^{(1)} = T_b$, and $T^{(2)} = T_t$.

Assume $\kappa_1 = \kappa_2$ and $\bar{T} = 1 - z$. In the non-dimensional form:

$$P_j^{-1} \frac{D\vec{v}_j}{Dt} = -P_j^{-1} \nabla \pi_j + R_j K_j \theta_j \mathbf{e}_z + R_j K_j \epsilon_j (2\bar{T}\theta_j + \theta_j^2)\mathbf{e}_z + \Delta \vec{v}_j,$$

$$\frac{D\theta_j}{Dt} - w_j = \Delta \theta_j, \quad \nabla \cdot \vec{v}_j = 0, \quad (j = 1, 2), \quad \vec{v}_j = (u_j, v_j, w_j)^T \tag{2.2}$$

where

$$R_j = \frac{\rho_0^{(j)} g \alpha_1^{(j)} (T^{(j)} - T_d) d^3}{\tilde{D}_j^3 \mu_1 \kappa_1}, \quad P_j = \frac{\nu_j}{\kappa_j}, \quad K_j = \tilde{D}_j^4, \quad \epsilon_j = \frac{\alpha_2^{(j)} (T^{(j)} - T_d) \tilde{D}_j}{\alpha_1^{(j)}}. \tag{2.3}$$

3. Boundary conditions

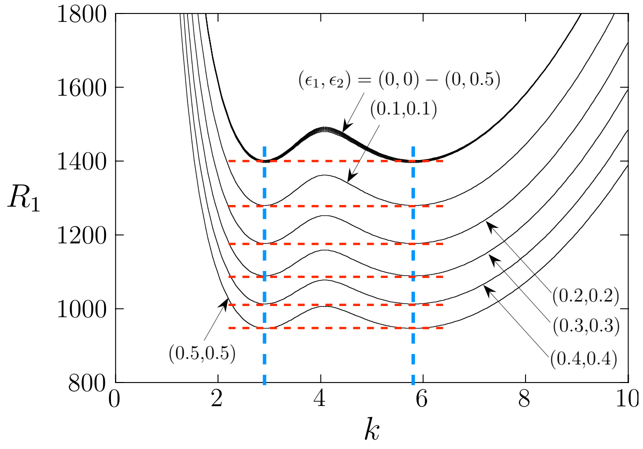
$$\vec{v}_1 = \vec{v}_2 = 0 \ \text{ at } \ z = 0, 1, 1 + D^{-1}, \quad \theta_1 = \theta_2 = 0 \ \text{ at } \ z = 0, 1 + D^{-1}. \tag{3.1}$$

On the dividing plate, the temperatures need to satisfy

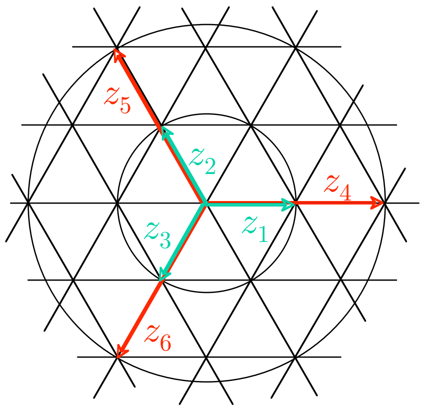
$$T_1^* = T_2^*, \quad \kappa_1 \frac{dT_1^*}{dz^*} = \kappa_2 \frac{dT_2^*}{dz^*} \ \text{ at } \ z^* = d, \tag{3.2}$$

which yield

$$\theta_1 = \frac{R_2}{R_1} \frac{D^4 \alpha_1^{(1)} \nu_2 \kappa_2}{\alpha_1^{(2)} \nu_1 \kappa_1} \theta_2 \equiv G\theta_2, \quad \frac{d\theta_1}{dz} = G \frac{d\theta_2}{dz} \ \text{ at } \ z = 1. \tag{3.3}$$



KF: Proc. R. Soc. A (2008) ***464*** *133–153*



$\psi(x,y,z,t) =$

$$z_1(t)\phi_1^{(1)}(z)\mathrm{e}^{ikx} + z_2(t)\phi_1^{(1)}\mathrm{e}^{ik(-\frac{1}{2}x+\frac{\sqrt{3}}{2}y)} + z_3(t)\phi_1^{(1)}\mathrm{e}^{ik(-\frac{1}{2}x-\frac{\sqrt{3}}{2}y)} + c.c.$$

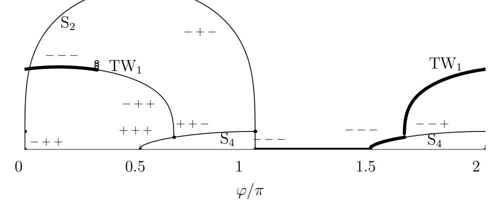
$$+z_4(t)\phi_2^{(1)}(z)\mathrm{e}^{2ikx} + z_5(t)\phi_2^{(1)}\mathrm{e}^{2ik(-\frac{1}{2}x+\frac{\sqrt{3}}{2}y)} + z_6(t)\phi_2^{(1)}\mathrm{e}^{2ik(-\frac{1}{2}x-\frac{\sqrt{3}}{2}y)} + c.c.$$

$$\dot{z}_1 = \sigma_1 z_1 + \delta_1 \bar{z}_2\bar{z}_3 + \beta_1 \bar{z}_1 z_4 + [\kappa_{11}|z_1|^2 + \kappa_{12}(|z_2|^2 + |z_3|^2)]z_1$$
$$+[\mu_{11}|z_4|^2 + \mu_{12}(|z_5|^2 + |z_6|^2)]z_1 + \nu_1 \bar{z}_1\bar{z}_5\bar{z}_6 + \xi_1 z_2 z_3 z_4$$
$$+\eta_1(\bar{z}_2 z_3 \bar{z}_6 + z_2 \bar{z}_3 \bar{z}_5),$$

$$\dot{z}_4 = \sigma_2 z_4 + \delta_2 \bar{z}_5\bar{z}_6 + \beta_2 z_1^2 + [\kappa_{21}|z_1|^2 + \kappa_{22}(|z_2|^2 + |z_3|^2)]z_4$$
$$+[\mu_{21}|z_4|^2 + \mu_{22}(|z_5|^2 + |z_6|^2)]z_4 + \nu_2 z_1 \bar{z}_2\bar{z}_3 + \xi_2(\bar{z}_3^2\bar{z}_5 + \bar{z}_2^2\bar{z}_6)$$

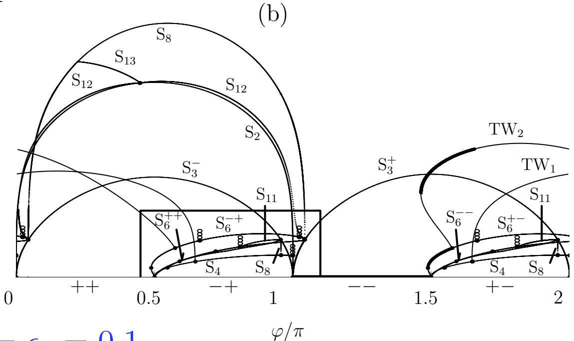
(a)

$I_2 = \mathbf{C}\{(1,0,0,0,0,0),(0,0,0,1,0,0)\}$ $\sigma_1 = \rho\cos\varphi, \quad \sigma_2 = \rho\sin\varphi$

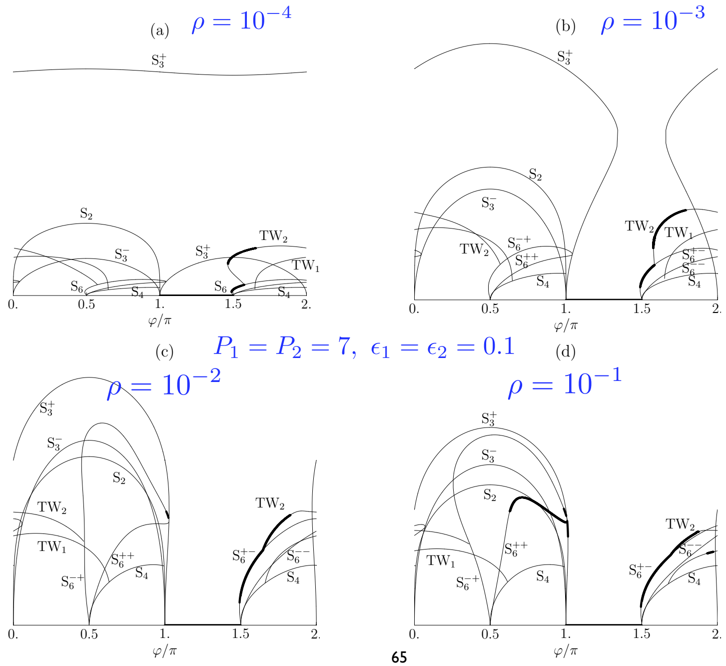


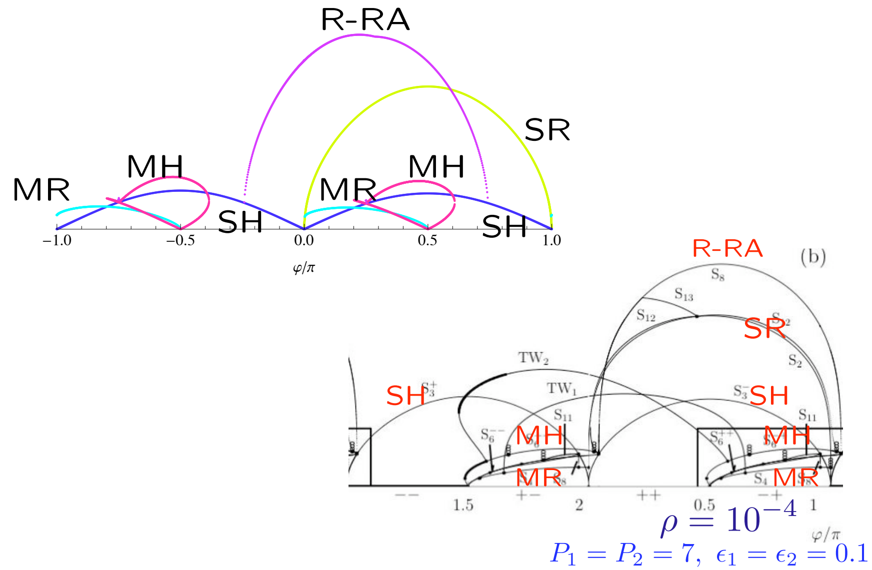
(b)

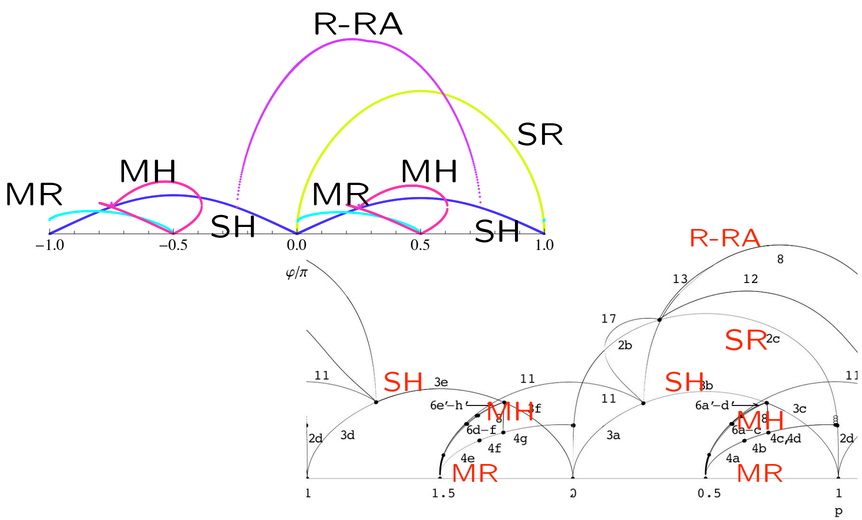
$\rho = 10^{-4}$



$P_1 = P_2 = 7, \ \epsilon_1 = \epsilon_2 = 0.1$







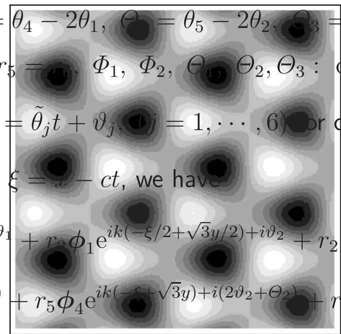
$P_1 = 150.76,\ P_2 = 7,\ \epsilon_1 = 0,\ \epsilon_2 = 0.1$

$\rho = 10^{-4}$

Traveling Waves TW2

$$z_j(t) = r_j(t)\,e^{i\theta_j(t)},\ j = 1, \cdots, 6,\quad \Phi_1 = \theta_1 + \theta_2 + \theta_3,\ \Phi_2 = \theta_4 + \theta_5 + \theta_6,$$

$$\Theta_1 = \theta_4 - 2\theta_1,\ \Theta_2 = \theta_5 - 2\theta_2,\ \Theta_3 = \theta_6 - 2\theta_3$$



We require $r_2 = r_3$, $r_5 = r_6$, $\Phi_1, \Phi_2, \Theta_1, \Theta_2, \Theta_3$: const.

$\dot\theta_j \Rightarrow$ const. $\Rightarrow \theta_j(t) = \tilde\theta_j t + \vartheta_j,\ (j = 1, \cdots, 6)$ for const. $\tilde\theta_j,\ \vartheta_j$.

Setting $\tilde\theta_1/k = c$ and $\xi = x - ct$, we have

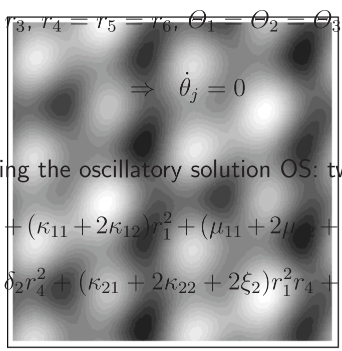
$$\hat{\psi} = r_1\phi_1 e^{ik\xi + i\vartheta_1} + r_2\phi_1 e^{ik(-\xi/2+\sqrt{3}y/2)+i\vartheta_2} + r_2\phi_1 e^{ik(-\xi/2-\sqrt{3}y/2)+i\vartheta_3}$$
$$+ r_4\phi_4 e^{2ik\xi + i(\vartheta_1+\Theta_1)} + r_5\phi_4 e^{ik(-\xi+\sqrt{3}y)+i(2\vartheta_2+\Theta_2)} + r_5\phi_4 e^{ik(-\xi-\sqrt{3}y)+i(2\vartheta_3+\Theta_2)}$$
$$+ c.c. + \text{higher order terms.}$$

TW_2 lie on the group orbit γz for $\gamma = (\tilde\theta_1 t, -\tilde\theta_1 t/2) \in T^2$.

Oscillatory Solution

Oscillatory Solution in $\mathbb{C}^6$

We require $r_1 = r_2 = r_3$, $r_4 = r_5 = r_6$, $\Theta_1 = \Theta_2 = \Theta_3 = 0$, and $\Phi_1 = 0$



$$\Rightarrow \quad \dot{\theta}_j = 0$$

The equations governing the oscillatory solution OS: two-dimensional for r_1, r_4

$$\dot{r}_1 = [\sigma_1 + \beta_1 r_4 + \delta_1 r_1 + (\kappa_{11} + 2\kappa_{12}) r_1^2 + (\mu_{11} + 2\mu_{12} + \nu_1) r_4^2 + (2\eta_1 + \xi_1) r_1 r_4] r_1,$$

$$\dot{r}_4 = \sigma_2 r_4 + \beta_2 r_1^2 + \delta_2 r_4^2 + (\kappa_{21} + 2\kappa_{22} + 2\xi_2) r_1^2 r_4 + (\mu_{21} + 2\mu_{22}) r_4^3 + \nu_2 r_1^3.$$

Structurally Stable Heteroclinic Cycles due to 1:2 Resonance under O(2)

- Armbruster, Guckenheimer & Holmes, *Physica* **29D** (1988) pp.257–282.
  Proctor & Jones, *J.Fluid Mech.* **188** (1988) pp.301–355.
  Porter & Knobloch, *Physica* **159D** (2001) pp.125–154.
  O(2) symmetric case under periodic boundary conditions.

$$\dot{z}_1 = \mu_1 z_1 + \alpha \bar{z}_1 z_2 + z_1(d_{11}|z_1|^2 + d_{12}|Z_2|^2),$$

$$\dot{z}_2 = \mu_2 z_2 + \beta z_1^2 + z_2(d_{21}|z_1|^2 + d_{22}|z_2|^2)$$

- Porter & Knobloch, *Physica* **201D** (2005) pp.318–344.
  Slightly broken symmetry: O(2) → SO(2)

$$\dot{z}_1 = (\mu_1 + i\epsilon\omega_1) z_1 + \alpha \bar{z}_1 z_2 + z_1(d_{11}|z_1|^2 + d_{12}|Z_2|^2),$$

$$\dot{z}_2 = (\mu_2 + i\epsilon\omega_2) z_2 + \beta z_1^2 + z_2(d_{21}|z_1|^2 + d_{22}|z_2|^2)$$

Structurally Stable Heteroclinic Cycles due to 1:2 Resonance under O(2)

- Cox, *Physica* **95D** (1996) pp.50–61.
  A long-wave PDE model + amplitude equations under O(2)

$$\frac{\partial \theta}{\partial t} = -\alpha\theta - \frac{R - R_0}{R_0}\frac{\partial^2 \theta}{\partial x^2} - a\frac{\partial^4 \theta}{\partial x^4} + b\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right)^3 + c\frac{\partial^2}{\partial x^2}\left(\frac{\partial \theta}{\partial x}\right)^2$$

- Mercader, Prat & Knobloch, *Int.J.Bifurcation and Chaos* **12** (2002) pp.2501–22.
  Rayleigh-Bénard convection without midplane reflection symmetry.
- Nore, Moisy & Quartier, *Phys.Fluids* (2005) **17** 064103.
  von Kármán swirling flow, laboratory experiment

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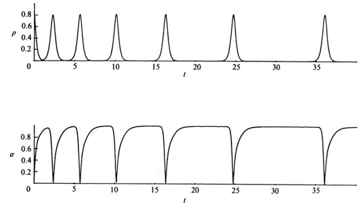
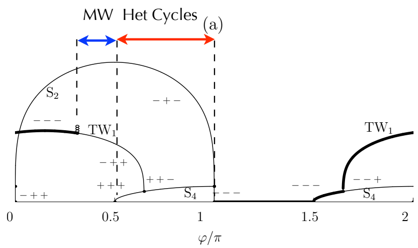
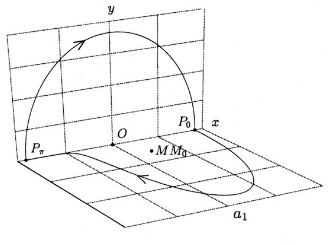
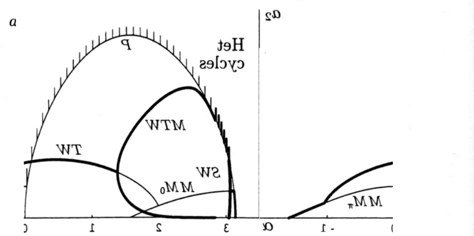


FIGURE 9. The approach to the homoclinic orbit, ρ and σ *vs.* time under system (5.1) with $a_1 = a_2 = 1$, $b_1 = b_2 = 0$, $\mu_1 = -0.8$, $\mu_2 = 1.0$, $\alpha = 5$.



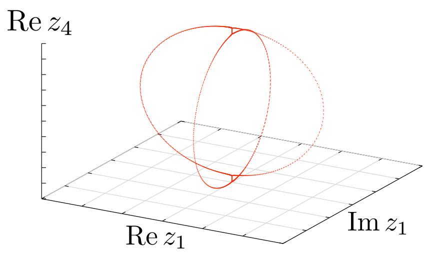


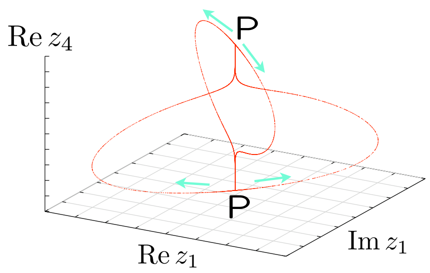
(a)



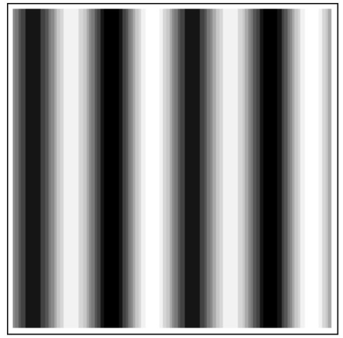
(b)

Fig. 19. (a) The structurally and asymptotically stable AGH cycle in the (a_1, x, y) variables for $\sigma = -1$, $d_{11} = -0.4$, $d_{12} = 1.6$, $d_{21} = -6$, $d_{22} = -0.5$ when $|\mu| = 0.05$ and $\alpha = 2.8$, where $\mu_1 = |\mu| \cos\alpha$, $\mu_2 = |\mu| \sin\alpha$. (b) The corresponding bifurcation diagram with α as the bifurcation parameter. After Porter and Knobloch [2001].

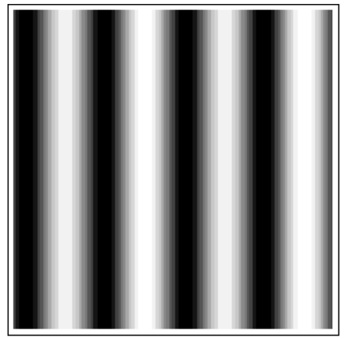




$(10^{-4}, \delta_2, \delta_3, 10^{-3}i, \delta_5, \delta_6), \quad \delta_i = 0$



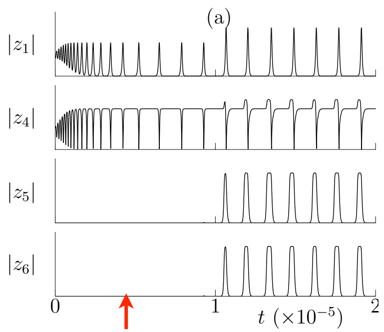
$\mathrm{TW}(I_2)$



$\mathrm{HC}(I_2)$

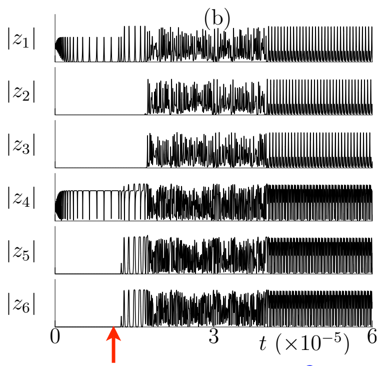
Breakdown of Nearly Heteroclinic cycles

$P_1 = 150.76, \; P_2 = 7, \; \epsilon_1 = 0, \; \epsilon_2 = 0.1$



$I_2 \longrightarrow I_4$

$\delta_n = O(10^{-24})$



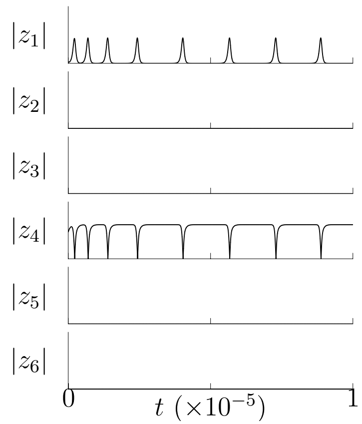
$I_2 \longrightarrow I_4 \longrightarrow \mathbb{C}^6$

$\delta_n = O(10^{-8})$

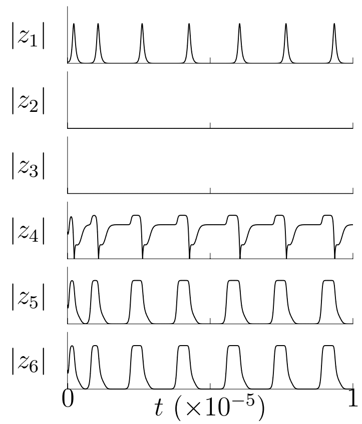
Nearly Heteroclinic Cycles in I 2 and I 4

$I_2 = \mathbf{C}\{(1,0,0,0,0,0), (0,0,0,1,0,0)\}$

$(10^{-4}, \delta_2, \delta_3, 10^{-3}i, \delta_5, \delta_6)$

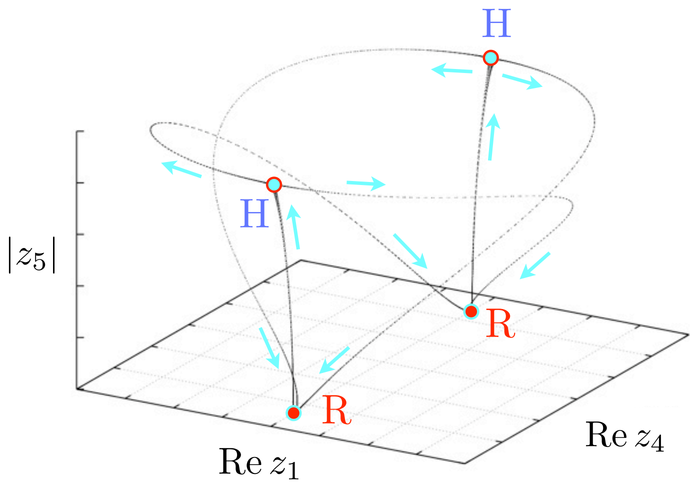


$\delta_n = 0$



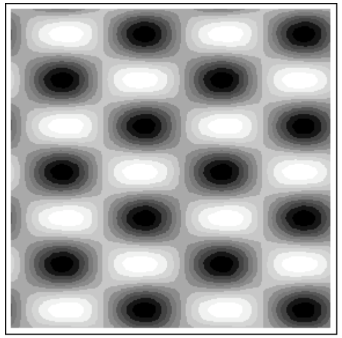
$\delta_n = O(10^{-24})$

$P_1 = 150.76, \; P_2 = 7, \; \epsilon_1 = 0, \; \epsilon_2 = 0.1$

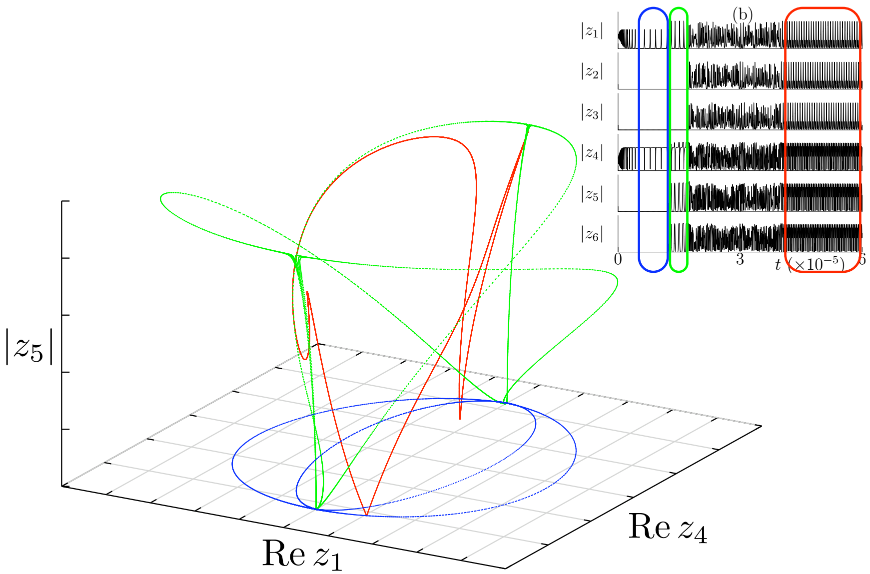


$\mathrm{HC}(I_4)$

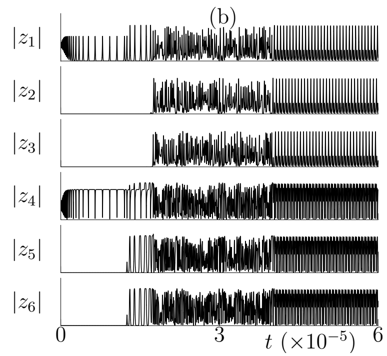
$(10^{-4}, \delta_2, \delta_3, 10^{-3}i, \delta_5, \delta_6),\ \ \delta_i = O(10^{-24})$



$\mathrm{HC}(I_4)$

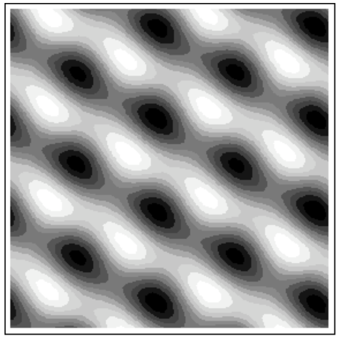


$\mathrm{HC}(I_2)$, $\mathrm{HC}(I_4)$, and $\mathrm{OS}(\mathbb{C}^6)$

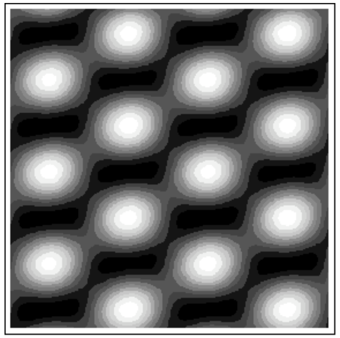


$\mathrm{Chaos}(\mathbb{C}^6)$ $\mathrm{OS}(\mathbb{C}^6)$

$$(10^{-4}, \delta_2, \delta_3, 10^{-3}i, \delta_5, \delta_6), \quad \delta_i = O(10^{-8})$$

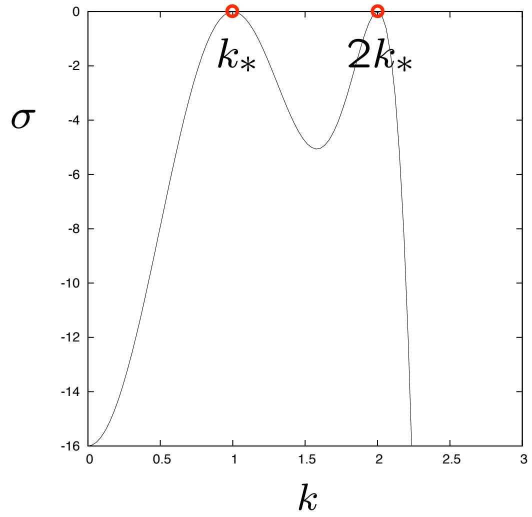


Chaos($\mathbb{C}^6$)



OS($\mathbb{C}^6$)

$$\sigma(k) = -(1-k^2)^2(4-k^2)^2$$



1 : 2 steady mode interaction

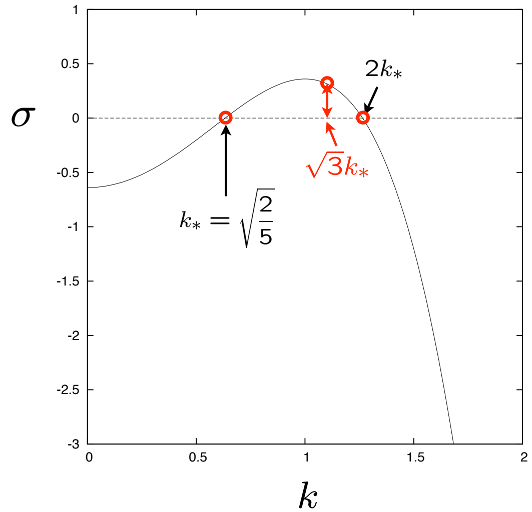
$$u(x,y) = z_{10}\,\mathrm{e}^{ikx} + z_{01}\,\mathrm{e}^{\frac{ik}{2}(x+\sqrt{3}y)} + z_{-1-1}\,\mathrm{e}^{\frac{ik}{2}(x-\sqrt{3}y)} + c.c.$$
$$+z_{20}\,\mathrm{e}^{2ikx} + z_{02}\,\mathrm{e}^{\frac{2ik}{2}(x+\sqrt{3}y)} + z_{-2-2}\,\mathrm{e}^{\frac{2ik}{2}(x-\sqrt{3}y)} + c.c. + \cdots$$

$$\dot{z}_{10} = \sigma_{10}z_{10} + 2\epsilon z_{11}z_{0-1} + 2\epsilon z_{20}z_{-10} + O(3),$$
$$\dot{z}_{20} = \sigma_{20}z_{20} + \epsilon z_{10}^2 + 2\epsilon z_{22}z_{0-2} + O(3).$$

Center manifold:

$$z_{jk} = h_{jk} = h_{jk}(z_{10}, z_{01}, z_{-1-1}, z_{20}, z_{02}, z_{-2-2}, \bar{z}_{10}, \bar{z}_{01}, \bar{z}_{-1-1}, \bar{z}_{20}, \bar{z}_{02}, \bar{z}_{22})$$

$$\sigma(k) = \frac{9}{25} - (1-k^2)^2$$



$1:2:\sqrt{3}$ steady-mode interaction

$$u(x,y) = z_{10}\,e^{ikx} + z_{01}\,e^{\frac{ik}{2}(x+\sqrt{3}y)} + z_{-1-1}\,e^{\frac{ik}{2}(x-\sqrt{3}y)} + c.c.$$

$$+z_{20}\,e^{2ikx} + z_{02}\,e^{\frac{2ik}{2}(x+\sqrt{3}y)} + z_{-2-2}\,e^{\frac{2ik}{2}(x-\sqrt{3}y)} + c.c.$$

$$+z_{21}\,e^{\frac{\sqrt{3}ik}{2}(\sqrt{3}x+y)} + z_{-11}\,e^{\frac{\sqrt{3}ik}{2}(-\sqrt{3}x+y)} + z_{-1-2}\,e^{-\sqrt{3}iky} + c.c. + \cdots$$

$$\Downarrow$$

$$\dot{z}_{10} = \sigma_1 z_{10} + 2\epsilon(z_{11}z_{0-1} + z_{20}z_{-10} + z_{21}z_{-1-1} + z_{1-1}z_{01}) + O(3),$$

$$\dot{z}_{20} = \sigma_2 z_{20} + 2\epsilon(z_{10}^2/2 + z_{22}z_{0-2} + z_{21}z_{0-1} + z_{1-1}z_{11}) + O(3),$$

$$\dot{z}_{21} = \sigma_{\sqrt{3}} z_{21} + 2\epsilon(z_{10}z_{11} + z_{20}z_{01} + z_{22}z_{0-1} + z_{12}z_{1-1}) + O(3).$$

Center-unstable manifold:

$$h_{jk} = h_{jk}(z_{10}, z_{01}, z_{-1-1}, z_{-10}, z_{0-1}, z_{11}, z_{20}, z_{02}, z_{-2-2},$$
$$z_{-20}, z_{0-2}, z_{22}, z_{21}, z_{12}, z_{-11}, z_{-2-1}, z_{-1-2}, z_{1-1})$$

1:$\sqrt{3}$ resonance $\cdots$ *damped KS*

Daumont, Kassner, Misbah & Valance: Phys.Rev.E **55** *(1997) pp.6902–6906.*

$$u_t = -\alpha u - \Delta u - \Delta^2 u + (\nabla u)^2$$

$$\sigma = 3/4 + q^2 - q^4$$