その粗視化がフラクタルとなるようなフラクタルの存在

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def. $(\boldsymbol{X}, \boldsymbol{\tau})$ **: self-similar**

i) $^{\exists}$**metric** $\boldsymbol{d}$ **s.t.** $\boldsymbol{\tau} = \boldsymbol{\tau_d}$

ii) $^{\exists}\{\boldsymbol{f_j} : (\boldsymbol{X}, \boldsymbol{\tau_d}) \longrightarrow (\boldsymbol{X}, \boldsymbol{\tau_d})$, **contraction,**

$\boldsymbol{j} \in \overline{\boldsymbol{m}} (= \{1, \cdots, \boldsymbol{m}\})\}$

s.t. $\displaystyle\bigcup_{j \in \overline{m}} \boldsymbol{f_j}(\boldsymbol{X}) = \boldsymbol{X}$

Proposition $^{\exists}$self-similar (Y,τ') s.t. $(Y,\tau') \simeq (X,\tau)$

$$\Longrightarrow (X,\tau) : \text{self-similar}$$

proof) $^{\exists}$ metric d on Y s.t. $\tau' = \tau_d$

$p_j : (Y,\tau_d) \to (Y,\tau_d), \;\; j = 1,\cdots,m$

$d(p_j(y),p_j(y')) \le \alpha_j(\eta)d(y,y'), \quad d(y,y') < \eta, \quad 0 \le \alpha_j(\eta) < 1$

$\cup_j p_j(Y) = Y, \quad h : (Y,\tau_d) \simeq (X,\tau),$

$\rho(x,x') = d(h^{-1}(x),h^{-1}(x')), \quad x,x' \in X,$

$$\tau = \tau_\rho$$

$$\begin{array}{ccc} Y & \xrightarrow{p_j} & Y \\ h\downarrow & & \downarrow h \\ X & \xrightarrow[q_j]{} & X \end{array}$$

$q_j = h \circ p_j \circ h^{-1} : (X,\tau_\rho) \to (X,\tau_\rho)$

(topologically conjugate to p_j)

$$\rho(q_j(x),q_j(x')) = d(h^{-1}(q_j(x)),h^{-1}(q_j(x')))$$
$$= d(p_j(h^{-1}(x)),p_j(h^{-1}(x'))) \le \alpha_j(\eta)d(h^{-1}(x),h^{-1}(x'))$$
$$= \alpha_j(\eta)\rho(x,x'), \;\; \rho(x,x') < \eta$$

$$\cup_j q_j(X) = \cup_j q_j(h(Y)) = h(\cup_j p_j(Y)) = h(Y) = X \;\; \square$$

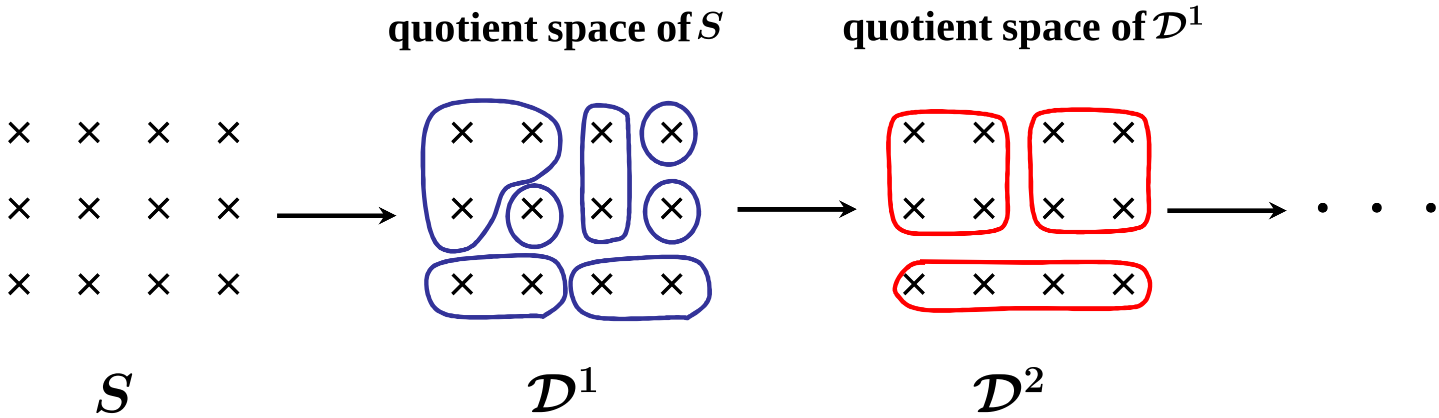
Coarse graining

$$\boldsymbol{Z} \longrightarrow \text{quotient space } \boldsymbol{Z/\!\sim} \text{ of } \boldsymbol{Z}$$

$$\uparrow$$

equivalence relation $\sim$

(Ariel Fernández, J. Phys. A: Math. Gen. 21 (1988) L295.)



位相的性質を保存するような粗視化列

$$S \simeq \mathcal{D}^1 \simeq \mathcal{D}^2 \simeq \cdots$$

$\simeq$: homeomorphic (**同相**)

$\boldsymbol{S}$ **: 0-dim, perfect, compact metric space**

$\Rightarrow {}^{\exists}$**sequence of quotient spaces**

$$\{\boldsymbol{S}, \boldsymbol{\mathcal{D}}^1, \boldsymbol{\mathcal{D}}^2, \cdots\}$$

s.t. $\boldsymbol{S} \simeq \boldsymbol{\mathcal{D}}^1 \simeq \boldsymbol{\mathcal{D}}^2 \simeq \cdots$

$$(\boldsymbol{X}, \boldsymbol{\tau}) : \textbf{0-dim} \overset{\text{def}}{=} {}^{\forall}\boldsymbol{x} \in \boldsymbol{X},\ {}^{\forall}\boldsymbol{U}(\boldsymbol{x}) \in \boldsymbol{\tau}$$

$$ {}^{\exists}\boldsymbol{u}(\boldsymbol{x}) \in \boldsymbol{\tau} \cap \mathfrak{S} \ \textbf{ s.t. } \boldsymbol{u}(\boldsymbol{x}) \subset \boldsymbol{U}(\boldsymbol{x})$$

$$(\boldsymbol{X}, \boldsymbol{\tau}) : \textbf{perfect} \overset{\text{def}}{=} {}^{\forall}\boldsymbol{x} \in \boldsymbol{X},\ \{\boldsymbol{x}\} \notin \boldsymbol{\tau}$$

$(\boldsymbol{X}, \tau_d)$: 0-dim, perfect, compact metric space

$\Rightarrow (\boldsymbol{X}, \tau_d) \simeq$ CMTS (self-similar)

$(\boldsymbol{X}, \tau_\rho)$: self-similar $\quad (\tau_d = \tau_\rho)$

self-similarとならないような0-dim, perfect, compactなmetric spaceは存在しない.

$\boldsymbol{S}$: **self-similar space**

$\Rightarrow \mathcal{D}^1, \mathcal{D}^2, \cdots$ **are all self-similar.**

A hierarchic structure of self-similar spaces

S : perfect, 0-dim, compact space

$$\cup_j f_j(S) = S \qquad (\{f_j\} - \text{self-similar})$$

$\mathcal{D}^1$: decomposition space of S

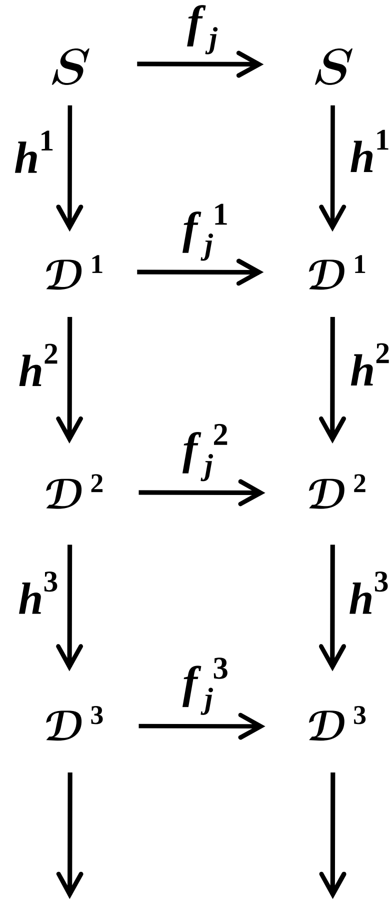
$$S \underset{h^1}{\simeq} \mathcal{D}^1, \qquad f_j^1 = h^1 \circ f_j \circ (h^1)^{-1}$$

$$\cup_j f_j^1(\mathcal{D}^1) = \mathcal{D}^1 \quad (\{f_j^1\} - \text{self-similar})$$

$\mathcal{D}^2$: decomposition space of $\mathcal{D}^1$

$$\mathcal{D}^1 \underset{h^2}{\simeq} \mathcal{D}^2, \qquad f_j^2 = h^2 \circ f_j^1 \circ (h^2)^{-1}$$
$$= (h^2 \circ h^1) \circ f_j \circ (h^2 \circ h^1)^{-1}$$

$$\cup_j f_j^2(\mathcal{D}^2) = \mathcal{D}^2 \quad (\{f_j^2\} - \text{self-similar})$$



(X, τ_d) : compact, $f_j : X \to X,\ j = 1, \cdots m,$

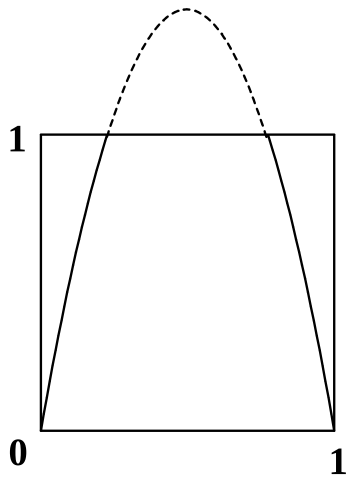
$d(f_j(x), f_j(x')) \le \alpha_j(\eta) d(x, x'),\ d(x, x') < \eta,\ 0 < \alpha_j(\eta) < 1,$

i) Each f_j is one to one, $\inf_{\eta>0} \alpha_j(\eta) > 0,$

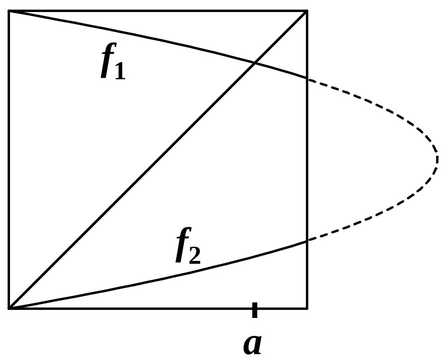
ii) $\cup_j \{x \in X \ ;\ x = f_j(x)\}$ is not a singleton,

iii) $\sum_j \inf_{\eta>0} \alpha_j(\eta) < 1.$

$\Longrightarrow {}^{\exists}$perfect, 0-dim, compact (S, τ_{d_S}) s.t. $\cup_j f_j(S) = S$ (self-similar)

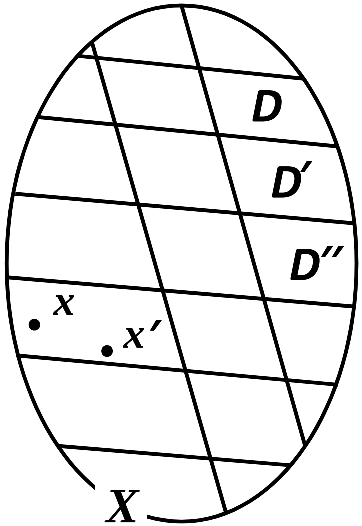


$\mu x(1-x),\ \mu > 4$



partition $\mathcal{D}$ of (X, τ) given

$$x \sim x' \overset{\text{def}}{=} {}^{\exists}D \in \mathcal{D} \quad \text{s.t.} \quad x, x' \in D$$



$$p : (X, \tau) \to X/\!\sim, \ x \mapsto C(x) = \{x' \in X \ ; \ x \sim x'\}$$

$$\tau_p = \{A \subset X/\!\sim \ ; \ p^{-1}(A) \in \tau\}$$

$$\tau(\mathcal{D}) = \{\mathcal{U} \subset \mathcal{D} \ ; \ \bigcup \mathcal{U} \in \tau\}$$

$$(X/\!\sim, \tau_p) = (\mathcal{D}, \tau(\mathcal{D}))$$

ex. $\mathcal{U} = \{D, D', D''\}$

$$\bigcup \mathcal{U} = D \cup D' \cup D'' \in \tau$$

ex. $f : (X, \tau) \to Y$, onto, $\qquad \mathcal{D}_f = \{f^{-1}(y) \subset X \ ; \ y \in Y\}$

A) $(\boldsymbol{Z}, \boldsymbol{\tau})$ **: perfect, 0-dim,** $\boldsymbol{T_0}$ **space**

$$^{\forall}\boldsymbol{n} \geq \mathbf{2},\ ^{\exists}\boldsymbol{Z}_1, \cdots, \boldsymbol{Z}_n \in \boldsymbol{\tau} \cap \mathfrak{S} - \{\phi\}$$
$$\text{s.t. } \boldsymbol{Z}_i \cap \boldsymbol{Z}_{i'} = \phi,\ \boldsymbol{i} \neq \boldsymbol{i}',\ \cup_i \boldsymbol{Z}_i = \boldsymbol{Z}$$

$$^{\forall}\boldsymbol{n} \geq \mathbf{2},\ ^{\exists}\boldsymbol{Z}_{i_1}, \cdots, \boldsymbol{Z}_{i_n} \in \boldsymbol{\tau} \cap \mathfrak{S} - \{\phi\}$$
$$\text{s.t. } \boldsymbol{Z}_{i_j} \cap \boldsymbol{Z}_{i_{j'}} = \phi,\ \boldsymbol{j} \neq \boldsymbol{j}',\ \cup_j \boldsymbol{Z}_{i_j} = \boldsymbol{Z}_i$$

$$\vdots$$

$(\boldsymbol{Z}_i, \boldsymbol{\tau}_{Z_i}),\ (\boldsymbol{Z}_{i_j}, \boldsymbol{\tau}_{Z_{i_j}}),\ \cdots$ **: perfect, 0-dim,** $\boldsymbol{T_0}$ **spaces**

(S, τ) : perfect, 0-dim, T_0 space

From A),

$^\exists S_1, \cdots, S_n \in \tau \cap \mathfrak{F} - \{\phi\}$ s.t. $S_i \cap S_{i'} = \phi,\ i \neq i',\ \cup_i S_i = S$

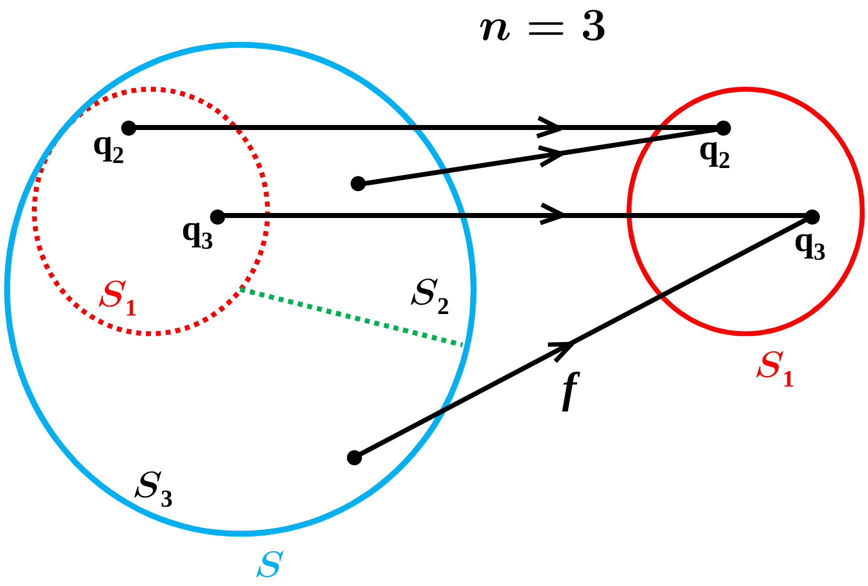
$q_2, \cdots, q_n \in S_1$

$f(x) = x$ for $x \in S_1$

$f(x) = q_2$ for $x \in S_2$

$\vdots$

$f(x) = q_n$ for $x \in S_n$



$f : (S, \tau) \longrightarrow (S_1, \tau_{S_1})$,

continuous, closed, onto, not one to one

$f : (S, \tau) \longrightarrow (S_1, \tau_{S_1})$,

continuous, closed, onto, not one to one

decomposition space $(\mathcal{D}_f, \tau(\mathcal{D}_f))$ **of** (S, τ)

$$\mathcal{D}_f = \{f^{-1}(x) \subset S \; ; \; x \in S_1\} \neq \{\{x\} \subset S \; ; \; x \in S\},$$

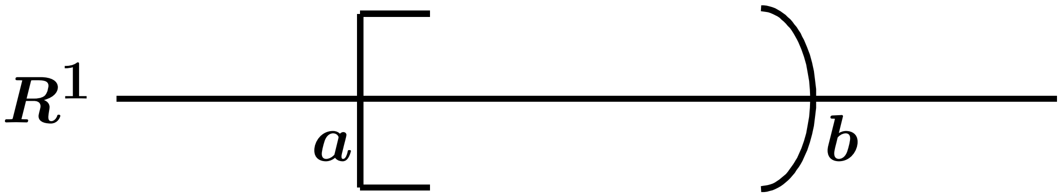
$$\tau(\mathcal{D}_f) = \{\mathcal{U} \subset \mathcal{D}_f \; ; \; \cup\,\mathcal{U} = \cup_{D \in \mathcal{U}} D \in \tau\}$$

$h : (S_1, \tau_{S_1}) \simeq (\mathcal{D}_f, \tau(\mathcal{D}_f))$, $x \longmapsto f^{-1}(x)$

$(\mathcal{D}_f, \tau(\mathcal{D}_f))$: **0-dim, perfect.**

Sorgenfrey line $(\boldsymbol{R}^1, \boldsymbol{\tau})$

base for $\boldsymbol{\tau}$; $\{\ [\boldsymbol{a}, \boldsymbol{b})\ ;\ \boldsymbol{a} < \boldsymbol{b}\}$



1^{st} countable, separable $\boldsymbol{T_2}$,

not metrizable, normal space

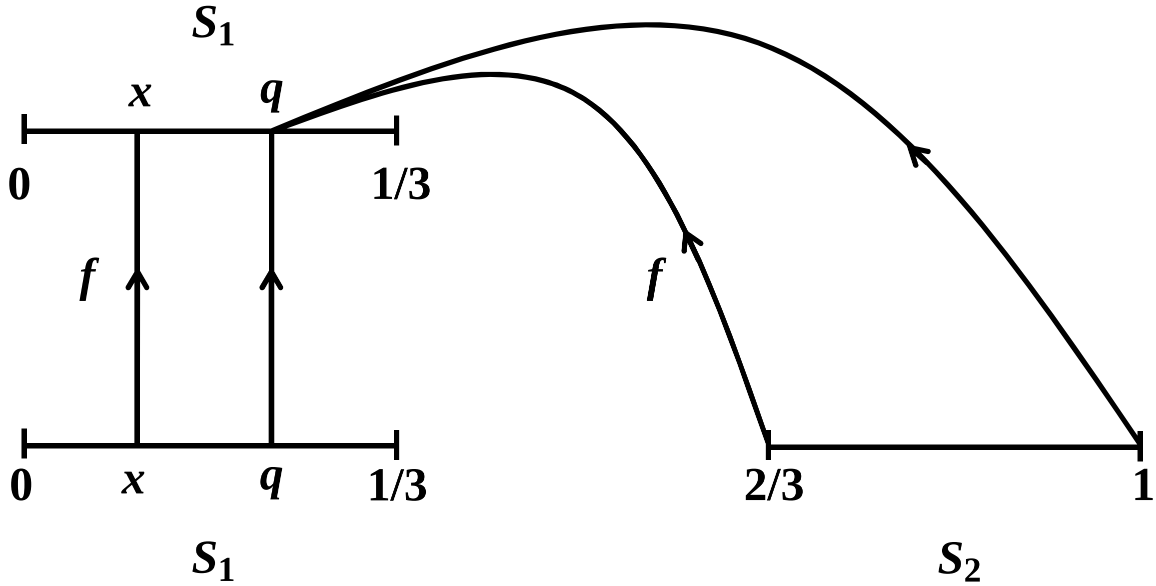
$$(\boldsymbol{S}, \boldsymbol{\tau}) = (\boldsymbol{S}, \boldsymbol{\tau}_d) : \textbf{0-dim, perfect, compact}$$

$$(\mathcal{D}_f, \boldsymbol{\tau}(\mathcal{D}_f)) \simeq (\boldsymbol{S}_1, \boldsymbol{\tau}_{d_{S_1}}) \simeq \textbf{CMTS}$$

$$(\boldsymbol{S}, \boldsymbol{\tau}_d) \simeq \textbf{CMTS}$$

$$\therefore (\boldsymbol{S}, \boldsymbol{\tau}_d) \simeq (\mathcal{D}_f, \boldsymbol{\tau}(\mathcal{D}_f))$$

$S = \text{CMTS}$ $\qquad f : S \longrightarrow S_1$, continuous, onto, closed, not one to one.



$S_1 = \text{CMTS} \bigcap [0, 1/3], \quad S_2 = \text{CMTS} \bigcap [2/3, 1]$

$\mathcal{D}_f = \{f^{-1}(x) \subset \text{CMTS}\,; x \in S_1\} = \{\{x\} \text{ for } x \in S_1 - \{q\}, \{q\} \bigcup S_2\}$

$h : (S_1, \tau_d) \simeq (\mathcal{D}_f, \tau(\mathcal{D}_f)), x \mapsto f^{-1}(x)$

ex. $y = \{x\}$, $y' = \{q\} \bigcup S_2$, ρ : metric on $\mathcal{D}_f$, $\tau(\mathcal{D}_f) = \tau_\rho$

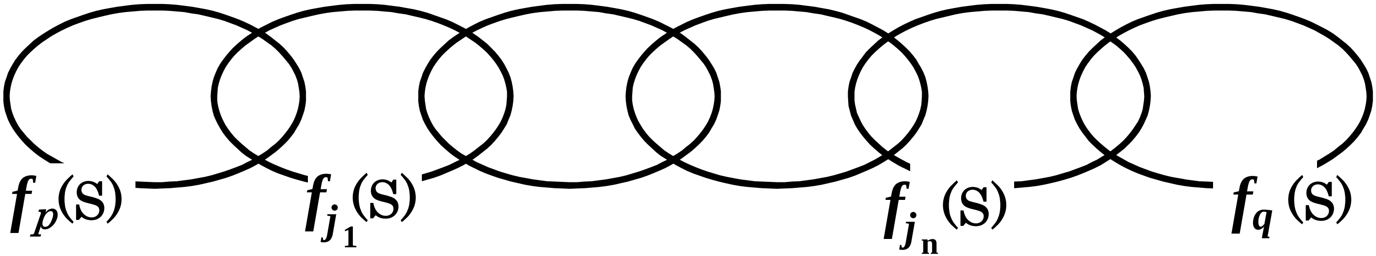
$$\rho(y, y') = d(h^{-1}(y), h^{-1}(y')) = d(x, q)$$

$\boldsymbol{S}$: **compact, self-similar metric space**

$^\forall p, q \in \overline{m} = \{1, \cdots, m\}$, $^\exists j_1, \cdots, j_n \in \overline{m}$ **s.t.**

$\{f_p(S), f_{j_1}(S), \cdots, f_{j_n}(S), f_q(S)\}$ **is a finite chain**

i.e.



$\Longrightarrow S$: **connected, locally connected.**

i.e. $^\exists f : [0, 1] \rightarrow S$, **continuous, onto**

from Hahn-Mazurkiewicz.

$(\boldsymbol{X}, \boldsymbol{\tau})$: compact, $\quad (\boldsymbol{Y}, \boldsymbol{\tau}')$: $\boldsymbol{T_2}$

$\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, continuous, onto

$$\Longrightarrow (\boldsymbol{Y}, \boldsymbol{\tau}') \simeq (\boldsymbol{\mathcal{D}_f}, \boldsymbol{\tau}(\boldsymbol{\mathcal{D}_f})), \ \boldsymbol{y} \mapsto \boldsymbol{f}^{-1}(\boldsymbol{y})$$

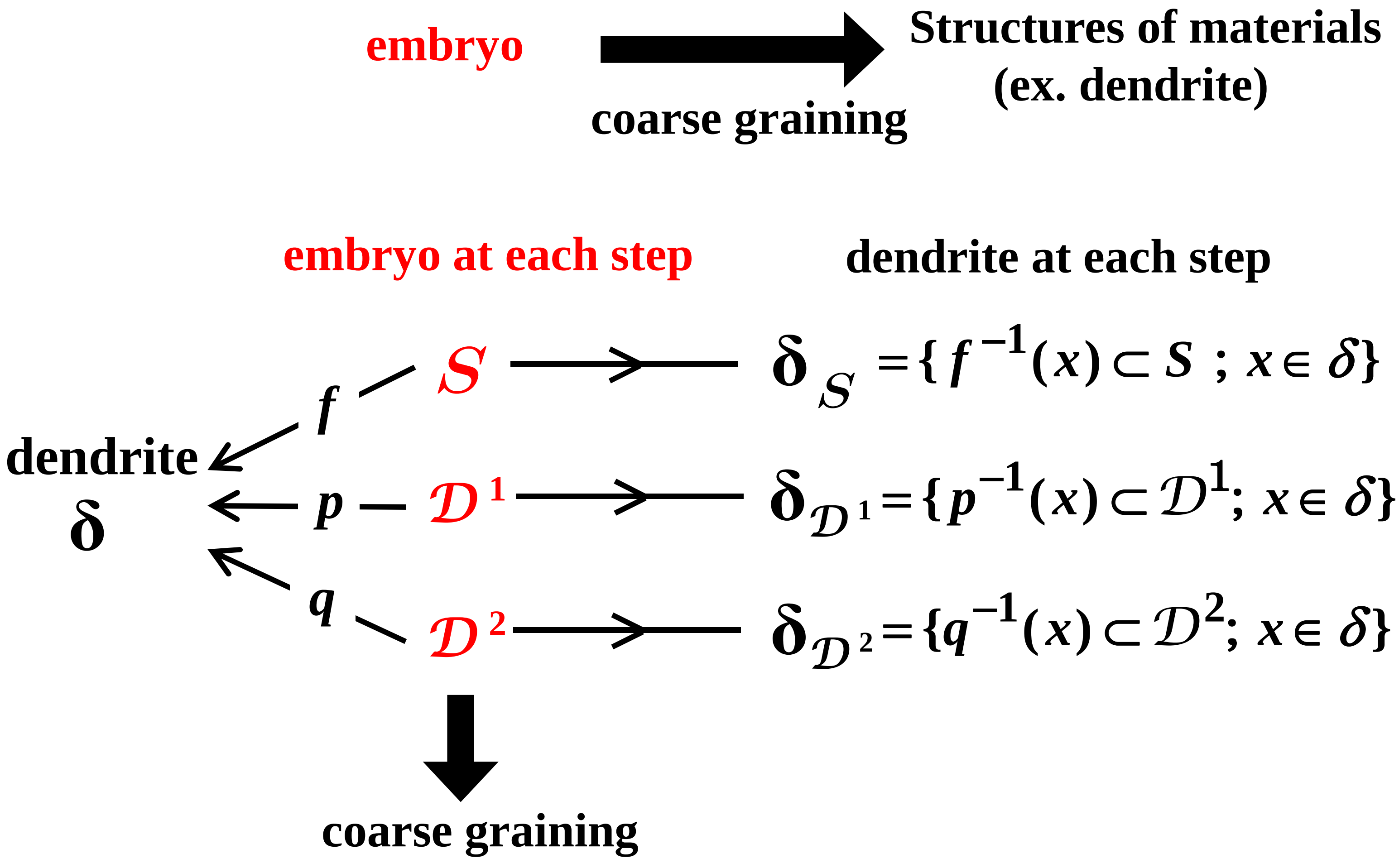
where $\boldsymbol{\mathcal{D}_f} = \{\boldsymbol{f}^{-1}(\boldsymbol{y}) \subset \boldsymbol{X} \ ; \ \boldsymbol{y} \in \boldsymbol{Y}\}$,

$$\boldsymbol{\tau}(\boldsymbol{\mathcal{D}_f}) = \{\boldsymbol{\mathcal{U}} \subset \boldsymbol{\mathcal{D}_f} \ ; \ \bigcup \boldsymbol{\mathcal{U}} \in \boldsymbol{\tau}\}$$

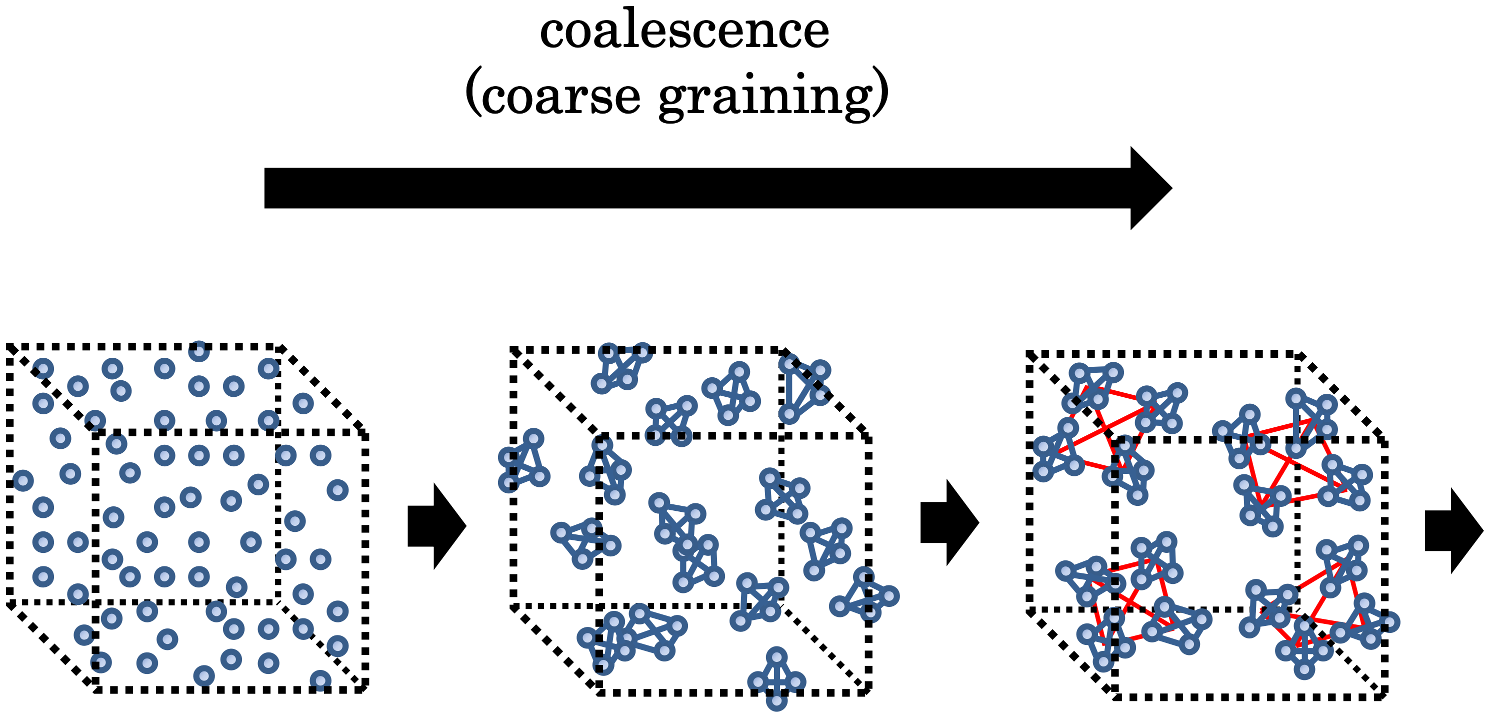
$(\boldsymbol{X}, \boldsymbol{\tau})$: 0-dim, perfect, compact $\mathbf{T_1}$ space

$(\boldsymbol{Y}, \boldsymbol{\tau_d})$: compact metric space

$$\Longrightarrow {}^{\exists}\boldsymbol{f} : \boldsymbol{X} \rightarrow \boldsymbol{Y}, \text{ continuous, onto}$$



embryo
coarse graining
Structures of materials
(ex. dendrite)
embryo at each step
dendrite at each step
dendrite
δ
f
p
q
S
$\mathcal{D}^1$
$\mathcal{D}^2$
$\delta_S = \{ f^{-1}(x) \subset S \ ; x \in \delta \}$
$\delta_{\mathcal{D}^1} = \{ p^{-1}(x) \subset \mathcal{D}^1; x \in \delta \}$
$\delta_{\mathcal{D}^2} = \{ q^{-1}(x) \subset \mathcal{D}^2; x \in \delta \}$
coarse graining


coalescence
(coarse graining)

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