1 次元非対称排他過程とノイズあり Burgers 方程式

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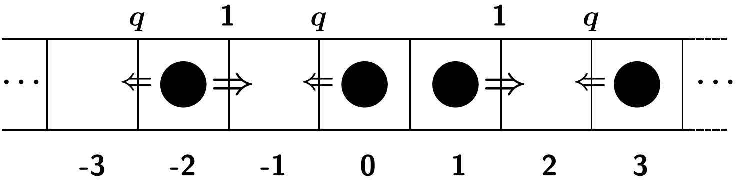
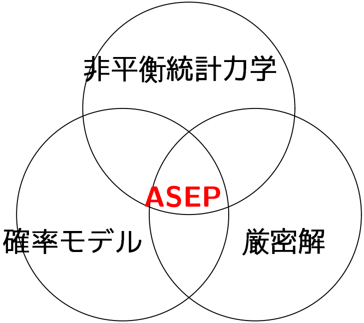
30 Jul 2010

ASEP = asymmetric simple exclusion process



- 各サイトは粒子がいるかいないかの 2 状態
- 微小時間 dt の間に各粒子は右隣のサイトに確率 dt で, 左隣のサイトに確率 qdt へホップしようとする.
- 行き先のサイトに既に粒子がいる場合は, 体積排除効果によりホッピングは起こらない.

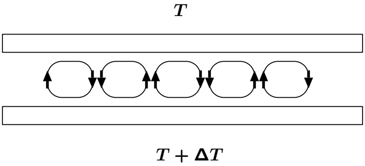
$q = 0$ の場合 $\cdots$ **Totally ASEP (TASEP)**


非平衡統計力学
ASEP
確率モデル
厳密解

非平衡模型としての ASEP

平衡から遠く離れた非平衡系

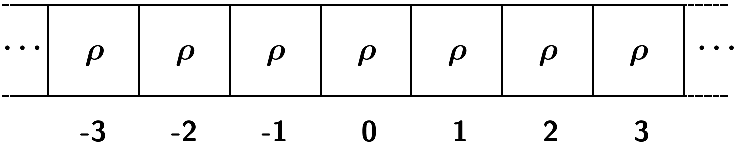
- 統計力学研究のフロンティア
- 流れの存在... 外界との物質・エネルギーのやりとり
- 長距離相関... 散逸構造 例: ベナール対流



- 定常的な性質と動的な性質

無限系の定常状態

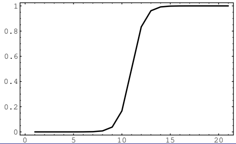
- **Bernoulli...** 各サイト独立に確率 ρ $(0 < \rho < 1)$ で粒子



カレント $J = (1 - q)\rho(1 - \rho)$

特に TASEP で $\rho = 1/2$ なら $J = 1/4$

- **Shock...** 密度が急激に変化する場所がある



確率モデルとしての ASEP

- **Spitzer** 1970
  Liggett 2 冊の本. 1985(Interacting particle systems), 1999(Stochastic interacting systems)
- Duality, Coupling, 流体力学極限 (ASEP$\Rightarrow$ Burgers eq)

$\cdots$	ρ	ρ	ρ	ρ	ρ	ρ	ρ	$\cdots$
	-3	-2	-1	0	1	2	3	

各粒子は rate $1-\rho$ でホップ $\Rightarrow$ Gaussian

$$\frac{X(t)-(1-\rho)t}{t^{1/2}} \sim N\left(0, \frac{1-\rho}{\rho}\right)$$

確率過程模型の一つの利点

形式的には量子系と等価

多粒子確率過程

	A		B	A		B

P = 系の確率分布　H = 遷移率行列

$$\frac{d}{dt}P(t) = -HP(t)$$

マスター方程式 = 虚時間シュレディンガー方程式

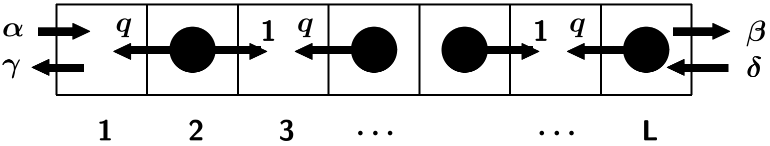
厳密に解ける模型としての ASEP

- ASEP は本質的に XXZ スピン鎖模型 (特別な値の虚数磁場がかかっていることに相当)
- (標準的な)1 次元 XXZ スピン鎖模型
  $H = -\sum_j \left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z\right]$
  - $\Delta = 0 \cdots$ 自由フェルミオン
  - Bethe ansatz, 量子群 $\cdots$ 固有値, 熱力学的性質, 相関関数
- ASEP は Bethe ansatz 等, XXZ 模型等のスピン鎖模型に対する手法およびそれとの直接の関係が一見明らかでない手法の両方で調べられてきた.

1 次元非平衡模型の厳密解

- 自由フェルミオン
  - 1 次元 Glauber ダイナミクス
    (1963 平衡への接近, 詳細釣合の原理)
  - 反応拡散モデル $A + A \to \phi$
- 可積分系の理論 (1980 年代-)
  - Yang-Baxter 方程式 (R R R = R R R)
  - 非エルミートなハミルトニアン $(H^\dagger \neq H)$
  - 可積分な 1 次元確率過程モデル $(H_{i,j} < 0, \sum_i H_{ij} = 0)$
  - 非線型相互作用, 平衡から遠く離れた系
    (1994 Alcaraz et al)

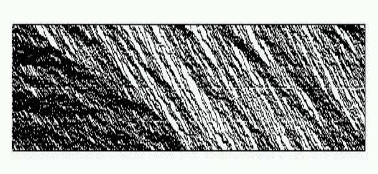
2. 開放系の定常状態



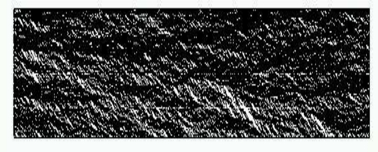
- 遷移率行列

$$H = \begin{bmatrix} \alpha & -\gamma \\ -\alpha & \gamma \end{bmatrix}_1 + \sum_{j=1}^{L-1} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & q & -1 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{j,j+1} + \begin{bmatrix} \delta & -\beta \\ -\delta & \beta \end{bmatrix}_L$$

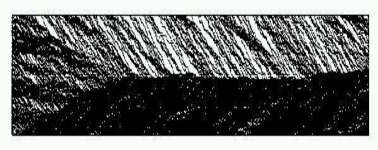
シミュレーション $(q = \gamma = \delta = 0)$



(A) α 小, β 大



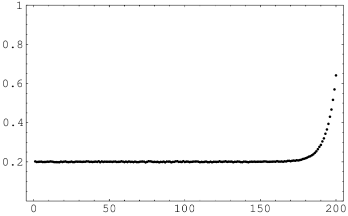
(C) α 大, β 大



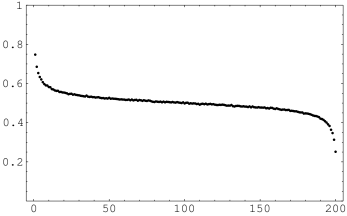
$\alpha = \beta$ 小

(B) α 大, β 小

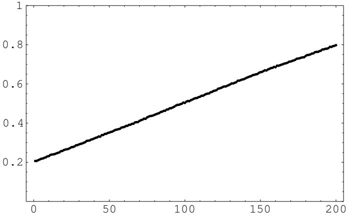
定常状態での密度プロフィール



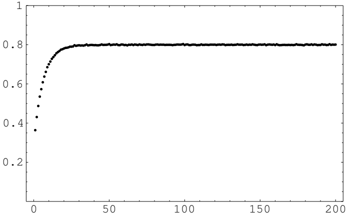
(A) 低密度相



(C) 流れ最大相



共存線



(B) 高密度相

行列の方法

1993 Derrida et al

$$\square \Leftrightarrow E \Leftrightarrow \tau_j = 0 \qquad \boxed{\bullet} \Leftrightarrow D \Leftrightarrow \tau_j = 1$$

定常状態

$$P(\tau_1, \tau_2, \ldots, \tau_L) = \frac{1}{Z_L} \langle W| \prod_{j=1}^{L} (\tau_j D + (1-\tau_j) E) |V\rangle$$

行列の満たすべき代数関係

$$DE - qED = D + E$$
$$\langle W|(\alpha E - \gamma D) = \langle W|$$
$$(\beta D - \delta E)|V\rangle = |V\rangle$$

一般に表現は無限次元の行列となる $\Rightarrow$ 長距離相関

境界条件による相転移

$q = \gamma = \delta = 0$ の場合 (1993 Derrida et al)

熱力学極限 $(L \to \infty)$ で

(A) 低密度相 $(\alpha < 1/2, \beta > \alpha)$

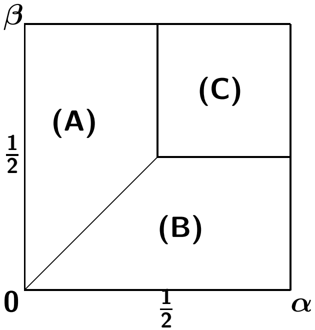
$$J = \alpha(1-\alpha)$$

(B) 高密度相 $(\beta < 1/2, \alpha > \beta)$

$$J = \beta(1-\beta)$$

(C) 流れ最大相 $(\alpha, \beta > 1/2)$

$$J = \frac{1}{4}$$



境界パラメータの値によってバルクでの性質が変わる!

パラメーター一般の場合

- $\gamma, \delta = 0$ の場合
  1999 S, 2000 Blythe Evans Colaiori Essler
  $q, \alpha, \beta, \gamma, \delta$ 一般の場合
  2004 Uchiyama S Wadati
- Askey-Wilson 多項式と呼ばれる直交多項式と関係
  カレント $J = \frac{Z_{L-1}}{Z_L}$

$$Z_L = \langle W|(D+E)^L|V\rangle$$
$$= \int_C \frac{dz}{4\pi i}\frac{((1+z)(1+z^{-1})/(1-q))^L(z^2,z^{-2};q)}{(az,a/z,bz,b/z,cz,c/z,dz,d/z;q)}$$

ただし
$(a;q) = \prod_{j=0}^{\infty}(1-aq^j), (a_1,\ldots,a_k;q) = \prod_{j=1}^{k}(a_j;q)$,
a, b, c, d は $\alpha, \beta, \gamma, \delta, q$ から定義

Askey Scheme

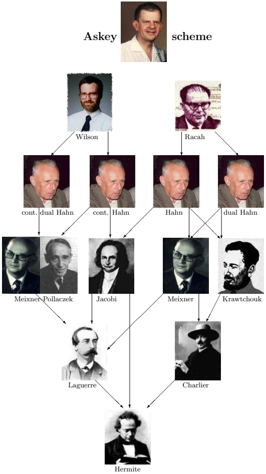


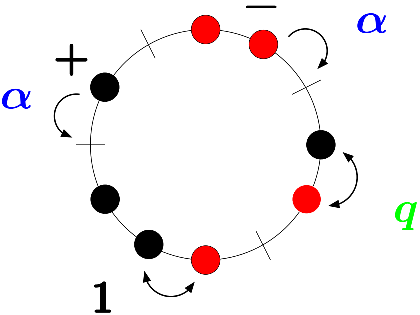
Figure 8: Askey scheme

(from Temme, “the Askey scheme and me”, 1968-2005)

3. 2成分 ASEP における凝縮現象

1998 Arndt Heinzel Rittenberg

$$
\begin{array}{lcll}
+\,0 & \rightarrow & 0\,+ & \text{with rate} \quad \alpha \\
0\,- & \rightarrow & -\,0 & \text{with rate} \quad \alpha \\
+- & \rightarrow & -+ & \text{with rate} \quad 1 \\
-+ & \rightarrow & +- & \text{with rate} \quad q
\end{array}
$$

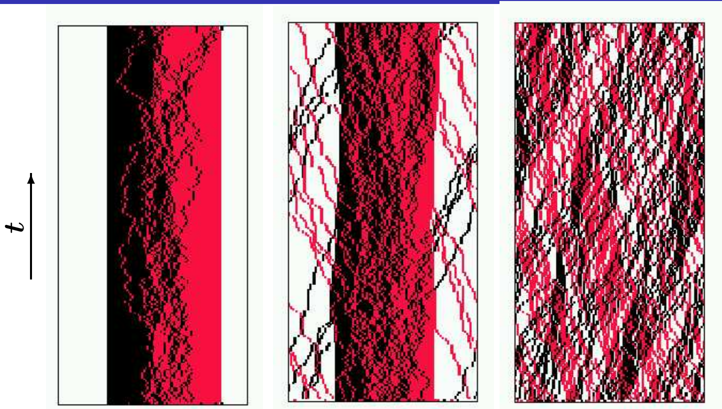


Periodic (L sites)

$$N_+ = N_-(= N)$$

$$\rho = N/L$$

シミュレーション



$q > 1$ $\quad$ $q < 1$, ρ **big** $\quad$ $q < 1$, ρ **small**

この凝縮は相転移現象か?

厳密解

2001 Rajewsky S Speer

系の大きさが無限の極限での粒子密度とカレントの関係

$$J(\xi) = (1-q)\xi\lambda(\xi)$$

$$\rho(\xi) = -\frac{\xi}{2}\frac{\partial}{\partial\xi}\ln\lambda(\xi)$$

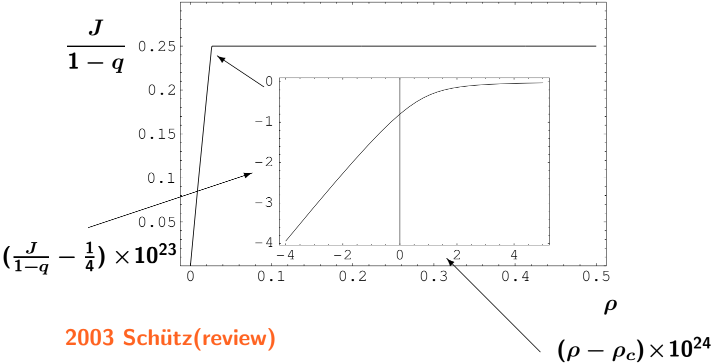
ここで $\lambda(\xi)$ は $f\left(y\left(\frac{1}{\lambda\xi}\right)\right) = \xi$ の解. ただし

$$f(y) = y\frac{(qy^2;q)_\infty(q;q)_\infty}{(ay;q)^2_\infty}\sum_{n=0}^{\infty}\frac{(ay;q)^2_n}{(qy^2;q)_n(q;q)_n}q^n, \quad a = -1+\frac{1-q}{\alpha}$$

$$y(x) = \frac{x-2-\sqrt{x^2-4x}}{2}, \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-}$$

$\alpha = 1, q = 0.9$ の場合の ρ-J 図

相転移は . . . 無さそう



2003 Schütz(review)

4. 時間に依存する性質

- カレント揺らぎ

$N(t)$: 時刻 t までに原点を越えた粒子数

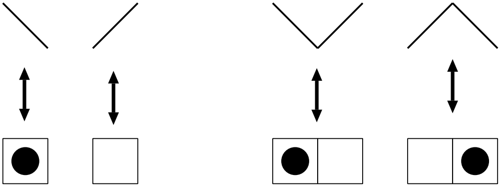
TASEP で密度 1/2 の場合, 平均 $\langle N(t) \rangle$

$$\lim_{t\to\infty} \frac{\langle N(t) \rangle}{t} = J = \frac{1}{4}$$

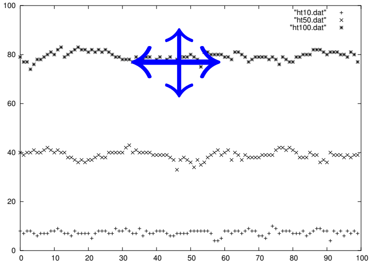
揺らぎは?

$$N(t) \sim \frac{t}{4} + O(t^{?})$$

界面成長モデルとの対応



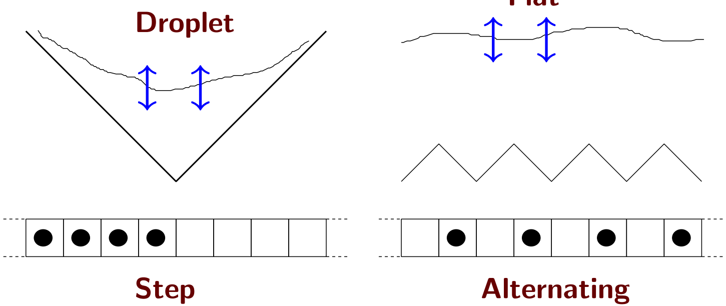
$N(t)$ は原点での界面高さに対応



カレントの揺らぎ

無限系の **Totally ASEP**(右のみに移動)

2 つの初期条件



以下, カレント $N(t)$ の分布について考える.

2000 Johansson(Young 盤の組合せ論)

$$\boxed{\lim_{t\to\infty} \mathrm{Prob}\left[\frac{\frac{t}{4}-N(t)}{2^{-4/3}t^{1/3}} < s\right] = F_2(s)}$$

ただし $F_2(s)$ は GUE Tracy-Widom 分布

$$F_2(s) = \det(1 - K_2\chi_s)$$

$$K_2(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y}$$

current of ASEP $\sim$ largest e.v. of RM

Step $\Leftrightarrow$ GUE

ランダム行列

GUE (Gaussian Unitary Ensemble)

$$A = \begin{bmatrix} u_{11} & u_{12}+iv_{12} & \cdots & u_{1N}+iv_{1N} \\ u_{12}-iv_{12} & u_{22} & \cdots & u_{2N}+iv_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1N}-iv_{1N} & u_{2N}-iv_{2N} & \cdots & u_{NN} \end{bmatrix}$$

$$\prod_{j=1}^{N} \frac{1}{\sqrt{\pi}} e^{-u_{jj}^2} \prod_{j<k} \frac{2}{\pi} e^{-2u_{jk}^2-2v_{jk}^2}$$

最大固有値 x_1 のスケールされた分布

$$\lim_{N\to\infty} \mathbb{P}\left[(x_1-\sqrt{2N})\sqrt{2}N^{1/6} < s\right] = F_2(s)$$

GOE (Gaussian Orthogonal Ensemble, $F_1(s)$)

$$A = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1N} \\ u_{12} & u_{22} & \cdots & u_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{2N} & u_{2N} & \cdots & u_{NN} \end{bmatrix}$$

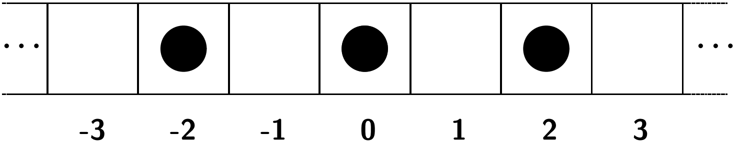
$$\prod_{j=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-u_{jj}^2/2} \prod_{j<k} \frac{1}{\sqrt{\pi}} e^{-u_{jk}^2}$$

GSE (Gaussian Symplectic Ensemble, $F_4(s)$)

時間依存版 $u_{jk} \longrightarrow u_{jk}(t)$

tGUE, tGOE, tGSE

Alternating の場合

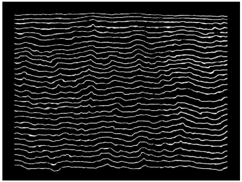


2000-2001 Baik et al, Prähofer et al (対称性のある組合せ論)

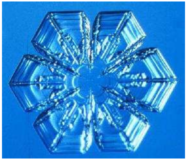
$$\boxed{\lim_{t\to\infty} \mathbb{P}\left[A_1(0) = \frac{\frac{t}{4} - N(t)}{2^{-4/3}t^{1/3}} < s\right] = F_1(s)}$$

1D surface growth

- Paper combustion, bacteria colony, crystal growth, liquid crystal turbulence (2010 Takeuchi Sano)
- Non-equilibrium statistical mechanics
- Stochastic interacting particle systems
- Exactly solvable models

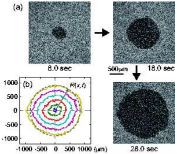


(Myllys et al)



(from Uwaha)

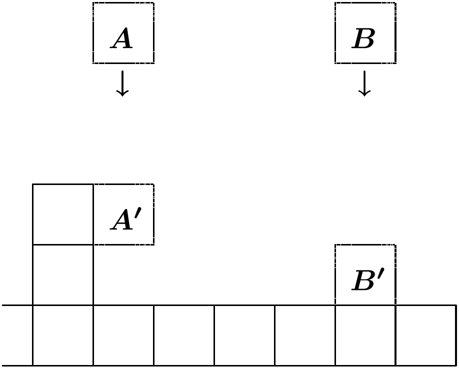
(Matsushita group)

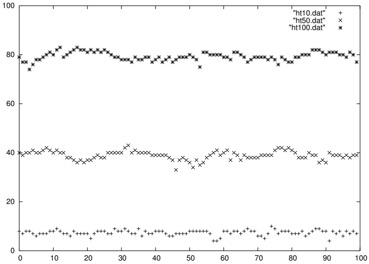


(Takeuchi Sano)

Simulation models

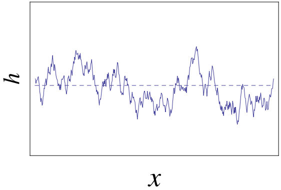
Ex: ballistic deposition model





Fluctuations of surface

$h(x,t)$: surface height at position x and at time t



Scaling (L: system size)

$$W(L,t) = \langle (h(x,t) - \langle h(x,t)\rangle)^2\rangle^{1/2}$$
$$= L^{\alpha}\Psi(t/L^z)$$

<u>For $t \to \infty$</u>

$$W(L,t) \sim L^{\alpha} \quad \Rightarrow \quad \Psi(y) \sim \text{Const}$$

<u>For $t \sim 0$</u>

$$W(L,t) \sim t^{\beta} \quad \Rightarrow \quad \Psi(y) \sim y^{\beta}$$

where $\alpha = \beta z$

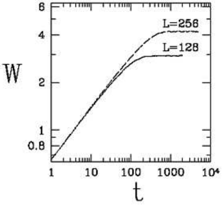


Figure 1. Interface width W versus time t for the RSO
(Ref. [11]) in $1+1$ dimensions, in two different lattice le

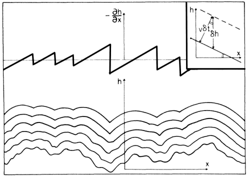
In many models, $\alpha = 1/2, \beta = 1/3$

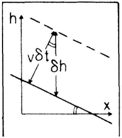
Kardar-Parisi-Zhang(KPZ) equation

1986 Kardar Parisi Zhang

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu\partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

where $\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$





$$\begin{aligned}\partial_t h &= v\sqrt{1+(\partial_x h)^2} \\ &\simeq v + (v/2)(\partial_x h)^2 + \dots\end{aligned}$$

Dynamical RG analysis $\rightarrow \alpha = 1/2, \beta = 1/3$ (**KPZ class**)

For $u(x,t) = \partial_x h(x,t)$,

$$\partial_t u = \nu\partial_x^2 u + \frac{\lambda}{2}\partial_x u^2 + \sqrt{D}\partial_x \eta(x,t)$$

(**noisy Burgers equation**)

Some comments

Warning!: There are several subtle points in the KPZ equation and its universality.

- The KPZ equation itself is ill-defined as it is due to the irregular behaviors of $h(x)$.
- Exponents are not enough to characterize the KPZ universality. Geometry dependence of distributions.
- The KPZ equation itself does not describe the KPZ universality class. One still has to take the scaling limit. Still the KPZ equation seems to be capturing some universal aspects of 1D surface growth.
  KPZ universality $\Leftrightarrow$ Universality of the KPZ equation
- There has been no exact solution for the KPZ equation.

Current distributions for ASEP with wedge initial conditions

2000 Johansson (TASEP) 2008 Tracy-Widom (PASEP)

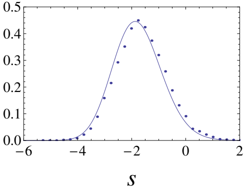
$$\mathcal{J}(0, t/(q-p)) \cong -\frac{1}{4}t + 2^{-4/3}t^{1/3}\xi_{\mathrm{TW}}$$

Here $\mathcal{J}(0,t)$ is the integrated current of ASEP at the origin and ξ_{TW} obeys the GUE Tracy-Widom distributions;

$$\mathbb{P}[\xi_{\mathrm{TW}} \le s] = \det(1 - P_s K_{\mathrm{Ai}} P_s)$$

where K_{Ai} is the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty d\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$



Current distributions for ASEP with flat initial conditions

Fluctuations are described by the GOE TW distributions.

These are the fluctuations of the KPZ universality class

(Rather, ASEP universality class?)

Questions

- What about the KPZ equation?
- Can we well-define the KPZ equation at all?
- Does the KPZ equation belong to the KPZ universality class, i.e., share the same distributions with ASEP, PNG?
- How universal is the KPZ equation?

Our Work

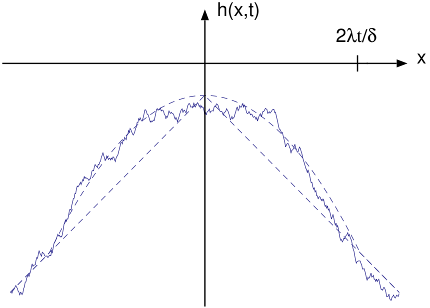
- The first exact result for the KPZ equation (for finite t)
- The KPZ equation is in the KPZ universality class
- The KPZ equation describes the growth with weak drive (Universality of the KPZ equation)
- Finite time correction, explaining $t^{-1/3}$ decay of mean

The same problem was studied independently by Amir Corwin Quastel arXiv:1003.0443

We consider the droplet growth and take the following narrow wedge initial conditions:

$$h(x,0) = -|x|/\delta\,, \quad \delta \ll 1$$

$$h(x,t) = \begin{cases} -x^2/2\lambda t & \text{for } |x| \le 2\lambda t/\delta\,, \\ -|x|/\delta & \text{for } |x| > 2\lambda t/\delta \end{cases}$$



Results

$$(\lambda/2\nu)h(x,t/2\nu) = -x^2/2t - \tfrac{1}{12}\gamma_t^3 + 2\log\alpha + \gamma_t\xi_t$$

Here $\gamma_t = 2^{-1/3}\alpha^{4/3}t^{1/3}$, $\quad \alpha = (2\nu)^{-3/2}\lambda D^{1/2}$.

The probability density of ξ_t

$$\rho_t(s) = \int_{-\infty}^{\infty} \gamma_t e^{\gamma_t(s-u)} \exp\left[-e^{\gamma_t(s-u)}\right] \times \Big(\det(1-P_u(B_t-P_{\mathrm{Ai}})P_u) - \det(1-P_uB_tP_u)\Big)du$$

where $P_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$, P_u is the projection onto $[u,\infty)$ and the kernel B_t is

$$B_t(x,y) = K_{\mathrm{Ai}}(x,y) + \int_0^{\infty} d\lambda (e^{\gamma_t\lambda}-1)^{-1} \times \Big(\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda) - \mathrm{Ai}(x-\lambda)\mathrm{Ai}(y-\lambda)\Big).$$

KPZ scaling limit

Let us write

$$\rho_t(s) = \frac{d}{ds}F_t(s) = \int_{-\infty}^{\infty} \gamma_t \mathrm{e}^{\gamma_t(s-u)} \exp\left[-\mathrm{e}^{\gamma_t(s-u)}\right] g_t(u)\mathrm{d}u$$

Here the first factor is the Gumbel probability density and

$$g_t(u) = \det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u).$$

In the $t \to \infty$ limit,

$$B_t \to K_{\mathrm{Ai}}, \quad g_t(s) \to F_{\mathrm{TW}}(s)u(s)$$

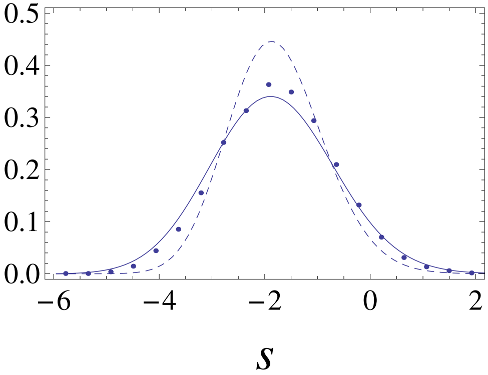
where $u(s) = \langle P_s \mathrm{Ai}, (1 - P_s K_{\mathrm{Ai}} P_s)^{-1} P_s \mathrm{Ai}\rangle$.

There is an identity $(\log F_{\mathrm{TW}})' = u$ and hence

$$\lim_{t\to\infty} F_t(s) = F_{\mathrm{TW}}(s)$$

The KPZ equation is in the KPZ universality class!

Finite time KPZ distribution and TW



—: exact KPZ density at $\gamma_t = 0.94$

− −: Tracy-Widom density

•: PASEP Monte Carlo at $q = 0.6$, $t = 1024$ MC steps

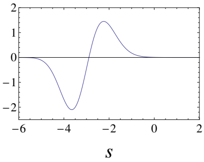
Finite time 1st order correction

We expand

$$g_t(u) = \det(1 - P_u(B_t - P_{\mathsf{Ai}})P_u) - \det(1 - P_u B_t P_u)$$

in $1/t$ by writing $B_t = K_{\mathsf{Ai}} + C_t$ and regarding C_t as a small perturbation. One arrives at

$$\begin{aligned} g_t(u) \cong\ & \rho_{\mathsf{TW}}(u) + 2\gamma_t^{-4}(\pi^4/15)\det_u(1 - K_{\mathsf{Ai}}) \\ & \times\big(\langle \mathsf{Ai}'', (1-K_{\mathsf{Ai}})^{-1}\mathsf{Ai}\rangle_u \langle \mathsf{Ai}', (1-K_{\mathsf{Ai}})^{-1}\mathsf{Ai}\rangle_u \\ & - \langle \mathsf{Ai}'', (1-K_{\mathsf{Ai}})^{-1}\mathsf{Ai}'\rangle_u \langle \mathsf{Ai}, (1-K_{\mathsf{Ai}})^{-1}\mathsf{Ai}\rangle_u\big) \end{aligned}$$



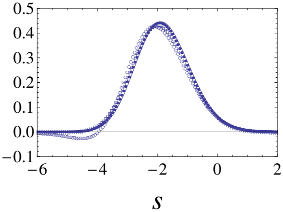
Large t expansion of ξ_t

$$\xi_t = \xi_{\mathsf{TW}} + 2^{1/3}\alpha^{-4}\left(\xi_{\mathsf{Gu}} + \log 2\alpha\right) t^{-1/3} + O(t^{-2/3})$$

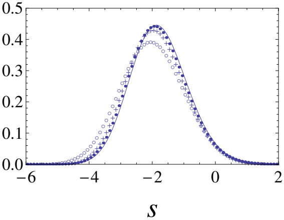
Explains the slow $t^{-1/3}$ decay of mean in TS experiment.

Figures for probability densities

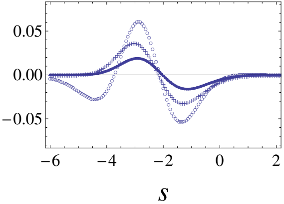
First order approximation



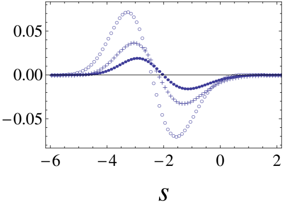
Evaluation of determinants



Different to TW



Difference to TW



$\gamma_t = 2\ (\circ)$

$\gamma_t = 5\ (+)$

$\gamma_t = 10\ (\bullet)$

Smoothing the noise

1997 Bertini Giacomin

The KPZ equation

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu\partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

with $\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$ is ill-defined as it is due to the irregular behaviors of $h(x)$.

How can one treat the equation?

We smoothen the noise η as

$$\langle \eta_\kappa(x,t)\eta_\kappa(x',t')\rangle = \varphi_\kappa * \varphi_\kappa(x-x')\delta(t-t')$$

with

$$\varphi_\kappa(x) = \kappa\varphi(\kappa x), \varphi \geq 0, \varphi \;\; \text{even}, \int dx\varphi(x) = 1$$

Cole-Hopf construction of the solution of KPZ equation

Cole-Hopf transform

$$Z_\kappa(x,t) = \exp[(\lambda/2\nu)h_\kappa(x,t)]$$

Z_κ **satisfies the linear equation**

$$\frac{\partial}{\partial t}Z_\kappa = \nu\frac{\partial^2}{\partial x^2}Z_\kappa + (\lambda\sqrt{D}/2\nu)\eta_\kappa Z_\kappa$$

Feynman path integral

$$Z_\kappa(x,t) = \mathbb{E}_x\Big(\exp\Big[(\lambda\sqrt{D}/2\nu) \times \int_0^t \mathrm{d}s\,\eta_\kappa(b(2\nu s), t-s)\Big] Z(b(2\nu t), 0)\Big),$$

where $\mathbb{E}_x(\cdot)$ denotes an average over the auxiliary Brownian motion $b(t)$, starting at x, $b(0) = x$, and with variance t.

$$\langle Z_\kappa(x,t)\rangle = (4\pi\nu t)^{-1/2}\exp\left[-x^2/4\nu t\right] \times \exp\left[\tfrac{1}{2}(\lambda\sqrt{D}/2\nu)^2\varphi_\kappa * \varphi_\kappa(0)t\right]$$

diverges. We use the Wick order

$$:Z_\kappa(x,t): = Z_\kappa(x,t)\exp\left[-\tfrac{1}{2}(\lambda\sqrt{D}/2\nu)^2\varphi_\kappa * \varphi_\kappa(0)t\right]$$

Now one can take the $\kappa \to \infty$ (KPZ) limit.

$$\lim_{\kappa\to\infty} :Z_\kappa(x,t): = :Z(x,t):$$

This satisfies

$$\langle :Z(x,t/2\nu): \rangle = \frac{1}{\sqrt{2\pi t}}\exp[-x^2/2t].$$

We *define* the KPZ height by

$$h(x,t) = (2\nu/\lambda)\log :Z(x,t):$$

KPZ as WASEP limit

1988 Gärtner, 2002 Bertini Giacomin

In PASEP, let us consider the space scale $\varepsilon^{-1}x$, the time scale $\varepsilon^{-2}t$ and set

$$p = \frac{1}{2}\left(1 - \beta\sqrt{\varepsilon}\right), \quad q - p = \beta\sqrt{\varepsilon}, \quad \tau = \frac{p}{q} \cong 1 - 2\beta\sqrt{\varepsilon}.$$

It is known that this corresponds to the KPZ equation.

One can utilize the results for PASEP to study the KPZ equation.

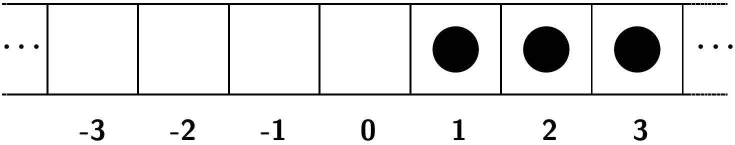
2009 Balázs Quastel Seppäläinen

1/3 exponent for stationary KPZ.

WASEP particle distribution

0-1 step initial conditions

(This corresponds to the narrow wedge initial conditions for KPZ equation.)



$x_m(t)$: the position of the mth particle from left

Result: The distribution of the particle position in WASEP

$$\lim_{\varepsilon\to 0}\mathbb{P}\Big(x_m(\varepsilon^{-2}t)-c_1\varepsilon^{-3/2}-\frac{c_2}{\gamma_t}\varepsilon^{-1/2}\log(2\beta\sqrt{\varepsilon})\le c_2 s\varepsilon^{-1/2}\Big)$$

$$=F_t(s).$$

Here

$$m = \sigma\beta t\varepsilon^{-3/2},$$
$$c_1 = (-1+2\sqrt{\sigma})\beta t\,, \quad c_2 = \sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}(\beta t)^{1/3}\,,$$
$$\gamma_t = 2\beta(\beta t)^{1/3}(\sqrt{\sigma}(1-\sqrt{\sigma}))^{2/3}\,, \quad 0<\sigma<1.$$

and

$$F_t(s) = 1 - \int_{-\infty}^{\infty} \exp[-\mathrm{e}^{\gamma_t(s-u)}] \times \Big(\det(1-P_u(B_t-P_{\mathrm{Ai}})P_u) - \det(1-P_uB_tP_u)\Big)\mathrm{d}u\,.$$

TW formula for PASEP

2008 Tracy Widom

$$\mathbb{P}\left(x_m(t/(q-p)) \le x\right) = \int_{\mathcal{C}_0} \prod_{k=0}^{\infty}(1-\mu\tau^k)\,\mathrm{det}(1+J(\mu))\frac{\mathrm{d}\mu}{\mu}$$

where

$$J(\mu;\eta,\eta') = \int_{\mathcal{C}_1} \frac{\varphi_\infty(\zeta)}{\varphi_\infty(\eta')}\frac{\zeta^m}{(\eta')^{m+1}}\frac{\mu f(\mu,\zeta/\eta')}{\zeta-\eta}\mathrm{d}\zeta$$

$$\varphi_\infty(\eta) = (1-\eta)^{-x}\mathrm{e}^{t(\eta/(1-\eta))}$$

$$f(\mu,z) = \sum_{k=-\infty}^{\infty}\frac{\tau^k}{1-\mu\tau^k}z^k$$

$\mathcal{C}_0$: a circle with around 0 and radius in $(\tau, 1)$

$\mathcal{C}_1$: a circle around 0 and radius in $(1, r/\tau)$

A question: Is the ASEP (or XXZ) a free fermion?

We investigate the limit of the kernel $J(\mu)$ as $\varepsilon \to 0$.
We write

$$\frac{\varphi_\infty(\zeta)\zeta^m}{\varphi_\infty(\eta')(\eta')^m} \times \frac{1}{\eta'(\zeta-\eta)} \times \mu f(\mu, \zeta/\eta') = Q_1 \times Q_2 \times Q_3 \,.$$

and take the scaling (we consider a shift of s later)

$$m = \sigma\beta t\varepsilon^{-3/2}\,, \quad x = c_1\varepsilon^{-3/2} + c_2 s\varepsilon^{-1/2}\,.$$

Saddle point analysis for Q_1 is the identical to the one of TW (usual KPZ scaling) with t replaced by $\varepsilon^{-3/2}$. The saddle point is given by

$$\xi = -c_4 = -2\beta(\beta t)^{1/3}(\sqrt{\sigma}(1-\sqrt{\sigma}))^{2/3}$$

With the substitutions

$$\eta \to \xi + c_3^{-1}\sqrt{\varepsilon}\eta\,, \quad \eta' \to \xi + c_3^{-1}\sqrt{\varepsilon}\eta'\,, \quad \zeta \to \xi + c_3^{-1}\sqrt{\varepsilon}\zeta\,.$$

where $c_3 = \sigma^{-1/6}(1-\sqrt{\sigma})^{5/3}(\beta t)^{1/3}$ **we find**

$$\lim_{\varepsilon\to 0} Q_1 = \exp[-\tfrac{1}{3}\zeta^3 + \tfrac{1}{3}(\eta')^3 + s(\zeta-\eta')]\,.$$

$$Q_2 = -\frac{c_3}{c_4\sqrt{\varepsilon}(\zeta-\eta)}\,.$$

The remaining is

$$Q_3 = \mu f(\mu, \zeta/\eta')$$

$$f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1-\mu\tau^k} z^k$$

Ramanujan summation formula

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(ax;q)_\infty (q/ax;q)_\infty (q;q)_\infty (b/a;q)_\infty}{(x;q)_\infty (b/ax;q)_\infty (b;q)_\infty (q/a;q)_\infty}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$$

$$(a;q)_n = (a;q)_\infty / (aq^n;q)_\infty$$

Using this one finds

$$f(\mu, z) = \frac{(\mu\tau z;\tau)_\infty (1/\mu z;\tau)_\infty (\tau;\tau)_\infty (\tau;\tau)_\infty}{(\tau z;\tau)_\infty (1/z;\tau)_\infty (\mu;\tau)_\infty (\tau/\mu;\tau)_\infty}$$

For $1 < |z| < \tau^{-1}$ it holds

$$\mu f(\mu, z) = \mu \frac{(\mu\tau z;\tau)_\infty (1/\mu z;\tau)_\infty (\tau;\tau)_\infty (\tau;\tau)_\infty}{(\tau z;\tau)_\infty (1/z;\tau)_\infty (\mu;\tau)_\infty (\tau/\mu;\tau)_\infty}$$
$$= \frac{1-\mu z}{(1-z)(1-\mu)} \prod_{n=1}^{\infty} \frac{(1-\tau^n)(1-\tau^n)}{(1-z\tau^n)(1-z^{-1}\tau^n)}$$
$$\times \prod_{n=1}^{\infty} \frac{(1-\mu z\tau^n)(1-(\mu z)^{-1}\tau^n)}{(1-\mu\tau^n)(1-\mu^{-1}\tau^n)}.$$

Since

$$z = \frac{\xi + c_3^{-1}\zeta\sqrt{\varepsilon}}{\xi + c_3^{-1}\eta'\sqrt{\varepsilon}} = 1 + (c_3 c_4)^{-1}(\eta' - \zeta)\sqrt{\varepsilon} + \mathcal{O}(\varepsilon)$$

we set

$$\mu f(\mu, 1 + \sqrt{\varepsilon} z) = Q_4 Q_5 Q_6 .$$

with

$$z = (c_3 c_4)^{-1}(\eta' - \zeta).$$

$$Q_4 = \frac{1-\mu(1+\sqrt{\varepsilon}z)}{-\sqrt{\varepsilon}z(1-\mu)} = \frac{1}{-\sqrt{\varepsilon}z}(1+\mathcal{O}(\sqrt{\varepsilon}))\,.$$

$$Q_5 = \prod_{n=1}^{\infty} \frac{(1-\tau^n)(1-\tau^n)}{(1-(1+\sqrt{\varepsilon}z)\tau^n)(1-(1+\sqrt{\varepsilon}z)^{-1}\tau^n)}$$

$$= -\Gamma_\tau(x)\Gamma_\tau(-x)(1+\sqrt{\varepsilon}z)^{-1}(\sqrt{\varepsilon}z)^2(1-\tau)^{-2}\,.$$

Here the q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}\,, \quad \text{when } |q|<1$$

and converges to the gamma function as $q \to 1$. Hence

$$\lim_{\varepsilon\to 0} Q_5 = \frac{\pi z/2\beta}{\sin(\pi z/2\beta)}\,.$$

$$Q_6$$

$$= \exp\left[\sum_{n=1}^{\infty}\left(\log\frac{1-\mu(1+\sqrt{\varepsilon}z)\tau^n}{1-\mu\tau^n}+\log\frac{1-(\mu(1+\sqrt{\varepsilon}z))^{-1}\tau^n}{1-\mu^{-1}\tau^n}\right)\right]$$

$$= \exp\left[\sum_{n=1}^{\infty}\left(\log\left(1-\sqrt{\varepsilon}z\frac{\mu\tau^n}{1-\mu\tau^n}\right)+\log\left(1+\frac{\sqrt{\varepsilon}z}{1+\sqrt{\varepsilon}z}\frac{\tau^n}{\mu-\tau^n}\right)\right)\right]$$

$$= \exp\left[\frac{z}{2\beta}\left(-\int_0^1 dy\frac{\mu}{1-\mu y}+\int_0^1 dy\frac{1}{\mu-y}\right)+\mathcal{O}(\sqrt{\varepsilon})\right]$$

$$= \exp\left[\frac{z}{2\beta}\log(-\mu)+\mathcal{O}(\sqrt{\varepsilon})\right]$$

The rescaled kernel of $J(\mu)$ is defined by

$$J^{\varepsilon}(\mu;\eta,\eta') = J(2\beta\mu\sqrt{\varepsilon};\xi + c_3^{-1}\eta\sqrt{\varepsilon},\xi + c_3^{-1}\eta'\sqrt{\varepsilon})c_3^{-1}\sqrt{\varepsilon}\,. \tag{1}$$

We have shown, it holds

$$\lim_{\varepsilon\longrightarrow 0} J^{\varepsilon}(\mu;\eta,\eta') = I(\mu;\eta,\eta')$$

where

$$I(\mu;\eta,\eta') = \int_{\Gamma_\zeta} \exp\Big[-\tfrac{1}{3}\zeta^3 + \tfrac{1}{3}(\eta')^3 + s(\zeta-\eta')\Big]\frac{1}{\zeta-\eta'} \times \frac{\pi}{\sin(\gamma_t^{-1}\pi(\eta'-\zeta))} e^{\gamma_t^{-1}(\eta'-\zeta)\log(-\mu)}\gamma_t^{-1}d\zeta$$

Hence

$$\lim_{\varepsilon\longrightarrow 0}\det(1+J^{\varepsilon}(\mu)) = \det(1+I(\mu))$$

Limit distributions

The rescaled distributions

$$F_t^{\varepsilon}(s) = \int_{(2\beta\sqrt{\varepsilon})^{-1}\mathcal{C}_0} \prod_{k=0}^{\infty}\left(1-2\beta\mu\sqrt{\varepsilon}\tau^k\right)\det(1+J^{\varepsilon}(\mu))\frac{1}{\mu}d\mu\,.$$

For the first factor in the integrand one sees

$$\lim_{\varepsilon\longrightarrow 0}\prod_{k=0}^{\infty}\left(1-2\beta\mu\sqrt{\varepsilon}\tau^k\right) = e^{-\mu}\,.$$

In the limit $\epsilon \longrightarrow 0$, we get

$$\lim_{\varepsilon\longrightarrow 0} F_t^{\varepsilon}(s) = \int_{\Gamma_\mu} e^{-\mu}\det(1+I(\mu))\frac{1}{\mu}d\mu =: \tilde{F}_t(s)$$

In fact $\tilde{F}_t(s) = F_t(s)$.

$$F_t(s) = \int_{\Gamma_\mu} e^{-\mu} \det(1 + I(\mu)) \frac{1}{\mu} d\mu$$
$$= 1 + \frac{1}{2\pi i} \int_0^\infty dv \frac{1}{v} e^{-v} \big(\det(1 - K_v^+) - \det(1 - K_v^-) \big) .$$

where

$$K_v^\pm(x, y) = \int_{\Gamma_\eta} d\eta \int_{\Gamma_\zeta} d\zeta \frac{\gamma_t^{-1}\pi}{\sin(\gamma_t^{-1}\pi(\eta - \zeta))} \exp\big[-\tfrac{1}{3}\zeta^3 + \tfrac{1}{3}\eta^3$$
$$+ \zeta y - \eta x + \gamma_t^{-1}(\eta - \zeta) \log v \pm i\gamma_t^{-1}\pi(\eta - \zeta) \big]$$
$$= B_t(x + u, y + u) \pm i(\pi/\gamma_t) \mathrm{Ai}(x + u) \mathrm{Ai}(y + u)$$

$$F_t(s) = 1 - \int_{-\infty}^{\infty} \exp[-e^{\gamma_t (s-u)}]$$
$$\times \big(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u) \big) du .$$

In the height picture

Now we want to translate the results for PASEP to KPZ equation

In terms of the surface height corresponding to PASEP,

$$h^{\varepsilon}(j,t) = -2\sum_{\ell=-\infty}^{j} \eta_\ell(t) + j$$

(η_ℓ is the occupation at site ℓ), the limit we obtained reads

$$\lim_{\varepsilon\to 0} \mathbb{P}\Big(\sqrt{\varepsilon}\beta h^{\varepsilon}(\lfloor \varepsilon^{-1}x\rfloor, \varepsilon^{-2}t) + \tfrac{1}{2}\beta^2 t\varepsilon^{-1} + (x^2/2t) - \log(2\beta\sqrt{\varepsilon}) \le \gamma_t s\Big) = F_t(s)\,,$$

with

$$\gamma_t = 2^{-1/3}(\beta^4 t)^{1/3}\,.$$

Centering

Remember the normalization of the KPZ

$$\langle :Z(x, t/2\nu):\rangle = \frac{1}{\sqrt{2\pi t}} \exp\Big[-\frac{x^2}{2t}\Big].$$

Let us set, for ASEP height $h(j)$,

$$f(j,t) = \mathbb{E}_t(e^{\vartheta h(j)}) \,.$$

ϑ is adjusted such that $e^{-2\vartheta} = \frac{p}{q}$. Then f is the solution of

$$\frac{d}{dt} f(j,t) = \frac{1}{\cosh\vartheta}\Big(\tfrac{1}{2} f(j+1,t) + \tfrac{1}{2} f(j-1,t) - f(j,t)\Big)$$
$$+\Big(\frac{1}{\cosh\vartheta} - 1\Big) f(j,t), \quad f(j,0) = e^{-\vartheta|j|} \,.$$

This can be used to normalize the WASEP limit to converge to Cole-Hopf solution of the KPZ equation.

Universality of the KPZ equation

- Our results are for the KPZ equation
- The KPZ equation is obtained as a weak asymmetric limit of the PASEP. Put differently, the KPZ equation describes the PASEP with small asymmetry for a time scale $(q-p)^{-4}$.
- The picture is expected to be universal for many surface growth models with weak drive.

Conclusions

- **Basic properties of ASEP have been explained.**
- **There are many techniques such as matrix product, random matrix, Bethe ansatz.**
- **The first exact solution for the KPZ equation**
- **Droplet growth, one-point height distributions**